Enriques曲面に魅せられて
2019年12月1日 関シンポジウム(奈良女大)

金銅誠之 (名大多元数理)

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§ Introduction F. Enriques 1896

C : 代数曲線 = compact Riemann 面 $g(C) = \dim H^0(C, \Omega^1_C)$ (genus) $g(C)=0 \Rightarrow C \cong \mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}}$

$g(C)$	0	1	≥ 2
C	$\mathbb{P}^1$	E elliptic curve	一般型
$\kappa(C)$ 小平次元	$-\infty$	0	1
$\mathrm{Bir}(C) = \mathrm{Aut}(C)$	PGL	E ⋊ 有限群	有限群

$\mathbb{P}^2(\mathbb{C}) \supset C = \{f_n(x,y,z)=0\}$ (斉次n次式) smooth

$n \leq 2 \Rightarrow C \cong \mathbb{P}^1$

$= 3 \Rightarrow C \cong$ elliptic curve

$\geq 4 \Rightarrow C$: gen. type

S : 代数曲面 $\dim H^0(S, \Omega^1_S) = \dim H^0(S, \Omega^2_S) = 0 \Rightarrow S$: rational ?

反例 : Enriques曲面

$\kappa(S)$	$-\infty$	0	1	2
S	有理曲面 ruled surface	アーベル曲面 Bielliptic K3 Enriques	楕円曲面 (honest)	一般型

$\mathbb{P}^3(\mathbb{C}) \supset S = \{f_n(x,y,z,w)=0\}$ smooth

$n \leq 3 \Rightarrow S$ rational

$n = 4 \Rightarrow$ K3

$n \geq 5 \Rightarrow$ 一般型

§① 例

§② モジュライ(周期)

§③ 自己同型群(+最近の話題, char = 2)

§① 例

S : 単連結でない。 $\exists \pi : \tilde{S} \xrightarrow{2:1} S$ universal cover ($\tilde{S}$: K3) $S \cong \tilde{S}/\langle\sigma\rangle$

σ : covering transf.

例① $\mathbb{P}^5 \ni (z_1, \cdots, z_6)$

$\tilde{S} : \sum_{i=1}^{6} z_i^2 = \sum \xi_i z_i^2 = \sum \xi_i^2 z_i^2 = 0$ $\sigma : = \begin{pmatrix} I_3 & 0 \\ 0 & -I_3 \end{pmatrix}$ 3次元族

$\{\xi_i\}$: $\mathbb{P}^1$の6点と思う。 $\mathbb{P}^1 \xleftarrow{2:1} C$ (6点で分岐) $g(C)=2$ $J(C)$ = Jacobian

$\mathrm{Km}(J(C)) \cong \tilde{S}$ (F. Klein)

例①′ (Godeaux)

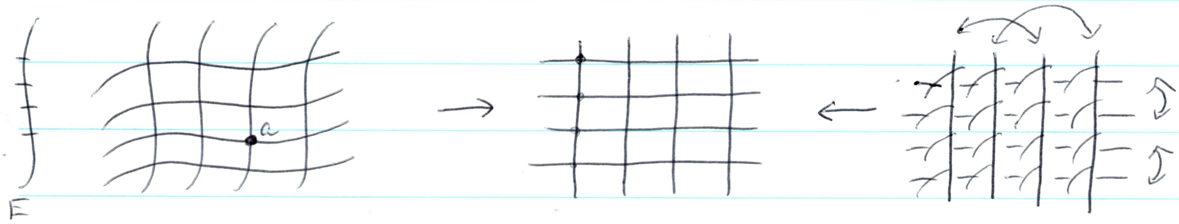
$Q_i(z_1, \cdots, z_6) = Q_i^+(z_1, z_2, z_3) + Q_i^-(z_4, z_5, z_6)$ (斉次2次式) $(i=1,2,3)$

$\tilde{S} : Q_1 = Q_2 = Q_3 = 0 \curvearrowleft \sigma$

$Gr(3,6) \times Gr(3,6) \curvearrowleft PGL$ 8 10次元族

(dim Gr(3,6) = 9, 9; 6 = dim{$\mathbb{P}^2$の2次式})

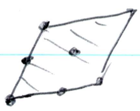
例② E, F: elliptic curve. $E\times F \longrightarrow E\times F/(-1,-1)$



16コの node　　　　$\mathrm{Km}(E\times F)$ = Kummer 曲面

2等分点

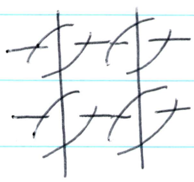
$(x,y) \downarrow (-x,-y)$　　$x^2=X,\ y^2=Y,\ xy=Z$　　$XY=Z^2$

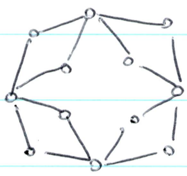


$(1,-1)$ は $\mathrm{Km}(E\times F)$ の involution ι を引き起こす。

$a=(a_1,a_2)\in E\times F$ 2-torsion　〃　t_a　〃

$\sigma=\iota\circ t_a$ 固定点をもたない involution





双対グラフ　　2次元族

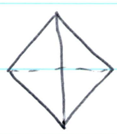
rank 10.

III型　　$H^2(S,\mathbb{Z})/\text{torsion}\cong\mathbb{Z}^{10}$

$$\sigma([z_1,z_2]) = \iota([z_1+a_1,\ z_2+a_2]) = [z_1+a_1,\ -z_2+a_2]$$
$$[z_1,z_2]=[-z_1,-z_2]$$

例③ (Enriques)

$\mathbb{P}^3\supset$ 6次式 $=0$　　tetrahedron の辺に沿って特異



$xyzt=0$

$K_{\tilde S}$: quadric で edge を通るもの

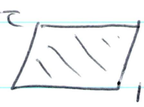
$2K_{\tilde S}$: quadric で edge を2重に通る $=\vee H_{ij}$

$\tilde S \searrow$

S: $x^2y^2z^2+x^2y^2t^2+x^2z^2t^2+y^2z^2t^2+\underbrace{Q}_{\text{斉次2次式}}\cdot xyzt=0$　　$\binom{3+2}{2}=10$ 次元族

§2. モジュライ(周期)

楕円曲線　$E=\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$　$\mathrm{Im}(\tau)>0$



$z\in\mathbb{C}$

dz: 平行移動で不変
⇝ ω_E: 正則一形式
$H^0(E,\Omega^1_E)\cong\mathbb{C}$

$\gamma_1,\gamma_2\in H_1(E,\mathbb{Z})$ basis

$$\mathrm{Im}\Big(\overbrace{\int_{\gamma_1}\omega_E\Big/\int_{\gamma_2}\omega_E}^{\tau(E)}\Big)>0$$

$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in GL(2,\mathbb{Z})$, $\cup$ $SL(2,\mathbb{Z})$　　$\tau(E)\longrightarrow \dfrac{a\tau(E)+b}{c\tau(E)+d}$

$SL(2,\mathbb{Z}) \curvearrowright H^+$ = 上半平面

$\{$楕円曲線$\}/\cong \quad \overset{1:1}{\longleftrightarrow}\quad H^+/SL(2,\mathbb{Z}) \cong \mathbb{C}$　(j-関数)

$j(\tau)=\dfrac{1}{\Delta(\tau)}$ ← discriminant

modular function

$H^1(E,\mathbb{Z})\cong\mathbb{Z}^2$

$H^1(E,\mathbb{C})\cong \underset{H^0(\Omega^1_E)}{H^{1,0}(E)}\oplus \underset{H^1(\mathcal{O}_E)}{H^{0,1}(E)}$　　Hodge分解

$\pi : \tilde{S} \to S$, $\circlearrowleft \sigma$

K3曲面の周期

$H^2(\tilde{S}, \mathbb{Z}) \cong \mathbb{Z}^{22}$

cup積 $\langle , \rangle : H^2(\tilde{S}, \mathbb{Z}) \times H^2(\tilde{S}, \mathbb{Z}) \to H^4(\tilde{S}, \mathbb{Z}) \cong \mathbb{Z}$

non-dg., symmetric bilin form, sgn $= (3,19)$, unimodular

格子 (lattice) $(H^2(\tilde{S}, \mathbb{Z}), \langle , \rangle) \cong L$: even, unimodular lattice, signature $(3,19)$

$$\cong U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1).$$

$$H^2(\tilde{S}, \mathbb{C}) \cong H^{2,0} \oplus H^{1,1} \oplus H^{0,2}, \quad H^{1,1} \cong \mathbb{C}^{20}, \quad H^{0,2} \cong \mathbb{C}$$

$H^{2,0} \cong H^0(\tilde{S}, \Omega^2) \cong \mathbb{C} \ni \omega_{\tilde{S}}$ 正則2形式.

$$\text{“}H_2(\tilde{S}, \mathbb{Z}) \longrightarrow \mathbb{C}\text{”} \in H^2(\tilde{S}, \mathbb{C}).$$
$$\gamma \longmapsto \int_\gamma \omega_{\tilde{S}}$$

指向曲面 H^+.

$\Updownarrow$

$$\begin{cases} \langle \omega_{\tilde{S}}, \omega_{\tilde{S}} \rangle = 0 \\ \langle \omega_{\tilde{S}}, \overline{\omega_{\tilde{S}}} \rangle > 0 \end{cases}$$

Enriques曲面の周期

$$\begin{array}{ccccc} & & \pi^* H^2(S,\mathbb{Z}) & & \\ & & \| & & \\ H^2(\tilde{S}, \mathbb{Z}) & \supset & H^2(\tilde{S}, \mathbb{Z})^+ & \oplus & H^2(\tilde{S}, \mathbb{Z})^- \\ \cong & & & & \\ L & \supset & L^+ & \oplus & L^- \end{array}$$

σ^*-eigen space

$\sigma^* \omega_{\tilde{S}} = -\omega_{\tilde{S}}$ $\quad (H^0(S, \Omega^2) = \{0\})$

符号 $(3,19)$ $\quad (1,9)$ $\quad (2,10)$.

$$\mathcal{D} := \{ [\omega] \in \mathbb{P}(L^- \otimes \mathbb{C}) \mid \langle \omega, \omega \rangle = 0, \ \langle \omega, \bar{\omega} \rangle > 0 \}. \qquad \dim = 10 \text{ の complex mfd.}$$

$\mathbb{P}(L^- \otimes \mathbb{C}) \cong \mathbb{P}^{11}$

$\langle z, z \rangle = z_1^2 - \sum z_i^2 + 2 z_{11} z_{12} = 0$

$\langle z, \bar{z} \rangle = |z_1|^2 - \sum |z_i|^2 + z_{11}\bar{z}_{12} + \bar{z}_{11} z_{12} > 0$

$$/\mathbb{C} \quad \langle z, z' \rangle = z_1 z_1' - \sum_{i=2}^{10} z_i z_i' + z_{11} z_{12}' + z_{11}' z_{12} = 0.$$

$$\mathrm{Im}(z_1)^2 - \sum \mathrm{Im}(z_i)^2 > 0.$$

$\alpha_S : H^2(S, \mathbb{Z})^- \xrightarrow{\sim} L^-$ isometry をとると $[(\alpha_S \otimes \mathbb{C})(\omega_{\tilde{S}})] \in \mathcal{D}$. $\quad \Gamma = O(L^-)$

$$\{\text{Enriques曲面}\}/\cong \ \overset{\alpha}{\hookrightarrow} \ \mathcal{D}/\Gamma \quad (\text{堀川})$$

Coble曲面 のところ

$\mathcal{H} = \{ [\omega] \mid \exists \delta \in L^-, \ \delta^2 = -2, \ \langle \omega, \delta \rangle = 0 \} \subset \mathcal{D}$ $\qquad \mathcal{H}/\Gamma$: $\mathrm{Im}(\alpha)$ の complement

- quasi-proj. (Baily-Borel)
- rational (Kondō)
- quasi-~~projective~~ affine (Borcherds) $\quad \exists \Phi$: 正則保型形式 で $(\Phi) = \mathcal{H}$.

(gen. super Kac-Moody alg の分母公式).

問 $\mathcal{D}/\Gamma$ $\begin{cases} \cdot \text{代数的構成} \\ \cdot \text{体 char } p > 0 \end{cases}$

§ 自己同型群

$\mathbb{P}^3 \supset X = \{f_4(x,y,z,t)=0\}$ 4次曲面 (⇒ K3曲面)

X : general ⇒ $\mathrm{Aut}(X) = \{1\}$

$X_F = \{x^4+y^4+z^4+t^4=0\}$ ↺ $(\mathbb{Z}/4\mathbb{Z})^3 \rtimes \mathfrak{S}_4$ $\qquad |\mathrm{Aut}(X_F)| = +\infty$

$\qquad = \left\langle \begin{pmatrix} 1 & & & \\ & \sqrt{-1}^a & & \\ & & \sqrt{-1}^b & \\ & & & \sqrt{-1}^c \end{pmatrix} \right\rangle$

F. Enriques 1907: S general (Enriques) ⇒ $|\mathrm{Aut}(S)| = \infty$. $\quad |\mathrm{Aut}|<\infty$ なる例?

G. Fano 1911: 例を構成 (VII型) $\quad \mathrm{Aut}(S) \cong \cancel{\mathfrak{S}_3}\ \mathfrak{S}_5$

1970年代 Piatetski-Shapiro, Shafarevich: K3曲面のTorelli

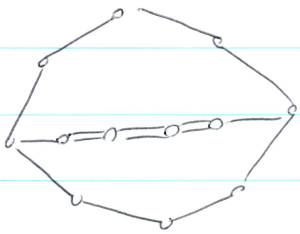
1978 Horikawa Torelli (Enriques曲面) $\qquad H^2(S,\mathbb{Z})/\mathrm{torsion}$

1983 Barth-Peters, S: gen. ⇒ $\mathrm{Aut}(S) \cong \mathrm{Ker}\,(O(L^{+}) \longrightarrow O(L^+/2L^+))/\pm 1$.

1984 V. Nikulin $|\mathrm{Aut}|<\infty$ なる Enriques曲面の"周期"を分類 (6種類)

'86 S.K 〃 を分類・構成 (I, II, …, VII)

例 I型



$\tilde{A}_1 \oplus \tilde{A}_7$, $\tilde{A}_1 \oplus \tilde{E}_7$, $\tilde{D}_8$, $\tilde{E}_8$

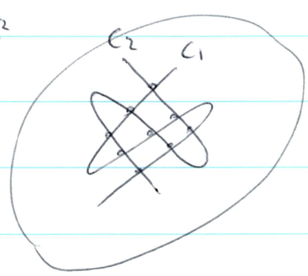
$A_n \quad D_n \quad E_6 \quad E_7 \quad E_8$

$\tilde{A}_n \quad \tilde{D}_n\ (n\geq 4) \quad \tilde{E}_6 \quad \tilde{E}_7 \quad \tilde{E}_8$

$I_{n+1} \quad I^*_{n-4} \quad \mathrm{IV}^* \quad \mathrm{III}^* \quad \mathrm{II}^*$

III

IV



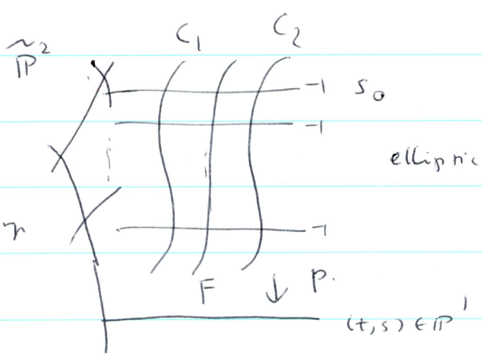
$C_1 : f_1 = 0$ ← 3次式

$C_2 : f_2 = 0$

blow up

$t f_1 + s f_2 = 0$

$(t,s) \in \mathbb{P}^1$



$$\mathrm{rank}\, H^2(\tilde{\mathbb{P}}^2, \mathbb{Z}) = \underset{\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}}{\mathrm{rk}\langle S_0, F\rangle} + \underbrace{\sum_{F:\ \text{a fiber}} (F\text{の既約成分の個数} - 1)}_{(\#)} + \mathrm{rank}\ \underbrace{\text{Mordell-Weil群}}_{\text{f.g. abelian gr}}$$

Enriques surface に対しても同様の式が成り立つ.

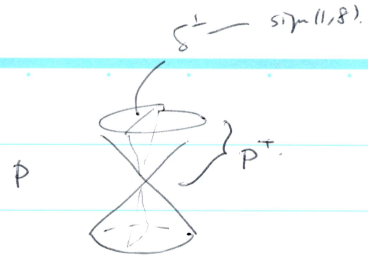
$$|\mathrm{Aut}(S)| < \infty \implies \text{"MW gr"} = 0\ (\text{torsion}) \iff (\#) = 8.$$

$L = H^2(S,\mathbb{Z})/\text{torsion} \cong \mathbb{Z}^{10}$ sign $=(1,9)$

$L\otimes\mathbb{R}$ $x_0^2 - x_1^2 - \cdots - x_9^2$

$\cup$

$P^+ := \{x \mid x^2>0\}^0$



$\Delta^+(S) = \{\delta\in L \mid \delta=[\mathbb{P}^1],\ \mathbb{P}^1\subset S\}$

S: gen. $\Rightarrow \Delta(S)=\emptyset$.

$\delta\in\Delta(S) \Rightarrow \delta^2=-2$.

$W(S)=\langle\{s_\delta \mid \delta\in\Delta(S)\}\rangle \subset O(L)$.

$s_\delta : x\longmapsto x+\langle x,\delta\rangle\delta$ reflection w.r.t. $\delta^\perp$

$\delta \longmapsto -\delta$.

$\circlearrowright$ P^+

$W(S)$の基本領域 $A(S) = \{x\in P^+ \mid \langle x,\delta\rangle>0\ \forall\delta\in\Delta(S)\}$ ample cone

$\mathrm{Im}(\mathrm{Aut}(S) \xrightarrow{\rho} O(L))$ $|\mathrm{Ker}(\rho)|<+\infty$. (Fact)

$\mathrm{Im}(\rho)\not\ni s_\delta$. $W(S)\cap\mathrm{Im}(\rho)=\{1\}$.

$[O(S):W(S)]<\infty \Rightarrow |\mathrm{Aut}(S)|<\infty$

"$\Updownarrow$" Vinberg.

$\forall$maximal ext. Dynkin diagram の rank $=8$. (#)

<u>最近の話題</u> $k=\bar{k}$, $\mathrm{char}(k)=p>0$ の場合. $p>2$ $\exists\ \tilde{S} \xrightarrow{2:1} S$ étale

<u>$p=2$</u> Bombieri–Mumford. $\exists\ \pi: \tilde{S} \xrightarrow{2:1} S$ canonical cover.
K3-like

3種類

$K_S\not\equiv 0$
$K_S^{\otimes 2}=0$
$b_2(S)=0$.

$\boxed{K_S: \text{numerically trivial},\ b_2(S)=0}$

- singular Enriques $K_S=0$ $H^1(\mathcal{O}_S)\cong k$ $\circlearrowleft$ Frobenius map (1:1) π: $\mathbb{Z}/2\mathbb{Z}$-cover
- classical Enriques 同じ空系 π: μ_2-cover
- supersingular Enriques $K_S=0$ $H^1(\mathcal{O}_S)\cong k$ $\circlearrowleft$ Frobenius (0-map) π: α_2-cover.

$\Downarrow$

$\{\alpha_{ij}\}$ cocycle

$\alpha_{ij}^2 = f_i - f_j$. $z_i^2=f_i$. $z_i-z_j=\alpha_{ij}$ より合せ

- $p>2$, $p=2$ singular $\mathbb{C}$上と同様 (Martin)

$\eta=\{df_i\}$ global 1-form

- $p=2$, supersingular, classical.
  松, Martin, K—.

$P\in\tilde{S}$ singularity $\Leftrightarrow$ $\pi(P)$: η の zero.

$c_2(S)=12$.

η has 12 zero (generically.)

- Vinberg
- $\tilde{S}$ の特異点 (Ekedahl, Shepherd-Barron)

Liedtke

hyperbolic lattice の reflection group
(sign $(1,n)$)

$L = \mathrm{II}_{1,25} = U \oplus \Lambda$ $\quad U = (\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$, $\quad \Lambda$ = Leech 格子 = even, unimodular, root $(=(-2)\text{-vector})$ を含まない
neg. def. rank = 24

$\quad \ni$
$x = (m, n, \lambda)$ $\quad m, n \in \mathbb{Z}$, $\lambda \in \Lambda$. $\quad x^2 = 2mn + \lambda^2$.

$\rho = (1, 0, 0)$. $\quad \rho^\perp \not\ni (-2)$-vectors. $\quad \left(\begin{array}{l} r = (m, n, \lambda). \\ \langle r, \rho \rangle = 0 \Leftrightarrow r = (m, 0, \lambda) \quad r^2 = \lambda^2 < -2 \end{array} \right)$

$r \in L$ $\quad r^2 = -2$ が $\underline{\text{Leech root}}$

$\Leftrightarrow \quad \langle r, \rho \rangle = 1 \quad \Leftrightarrow \quad r = (*, 1, \lambda)$.

$$\Delta = \{ r \mid r : \text{Leech root} \} \overset{1:1}{\longleftrightarrow} \Lambda$$
$$(*, 1, \lambda) \longleftarrow \lambda \in \Lambda$$

$\underline{\text{Thm}}$ (Conway) $\mathcal{C} := \{ x \in P^+(L \otimes \mathbb{R}) \mid \langle x, r \rangle > 0, \ \forall r \in \Delta \}$. 基本領域

$\pi : \tilde{S} \xrightarrow{2:1} S$ can. cover. $\quad \tilde{S}$: K3. $\quad \exists \ \mathrm{Pic}(\tilde{S}) \hookrightarrow L$ s.t. $\mathrm{Pic}(\tilde{S})^\perp \ni$ a Leech root
$\quad (1, n)$ $\qquad r_0 = (*, 1, \lambda_0)$

$\underline{\mathcal{C} \cap P^+(\tilde{S})}$ は finite polyhedron
$\uparrow$
幾何学的情報.

$r \in \Delta$. $\quad (r - r_0)^2 = (*, 0, \lambda - \lambda_0)^2 = (\lambda - \lambda_0)^2$.
$\quad = -4 - 2\underline{\langle r, r_0 \rangle}$
$\qquad \qquad \quad = 0, 1$.

$\underline{\text{Thm}}$ (Brandhorst – 島田)

$L^+ = U(2) \oplus E_8(2)$ の $\mathrm{II}_{1,25}$ への primitive な埋め込みは 丁度 17 通り.

埋め込み	1	2	3	…	7	8	9	10	11	12	13	14	15	16	17
# facets	12	12	20	…	$\boxed{20}$	$\boxed{20}$	$\boxed{40}$	$\boxed{40}$	$\boxed{40}$	$\boxed{40}$	40	$\boxed{96}$	$\boxed{96}$	96	?
Enriques surf.	I	II	III		VII	VII	MI	MI	MII	MII	MIII				
	/ℂ					p=2	/ℂ	p=2	/ℂ	p=2	/ℂ				

Mukai

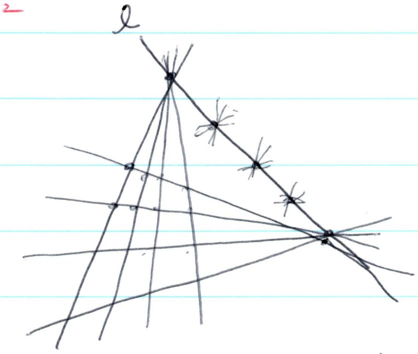
$\uparrow$ complement $\not\ni$ Leech roots

島田: コンピュータで存在を確認
sign K3

例 char $= 2$. $\mathbb{P}^2(\mathbb{F}_4)$ 21 lines 21 points. $\frac{4^3-1}{4-1} = 21$ $\ell = \mathbb{P}^1(\mathbb{F}_4) \ni 5$点

$(21)_5$-conf.

MII型

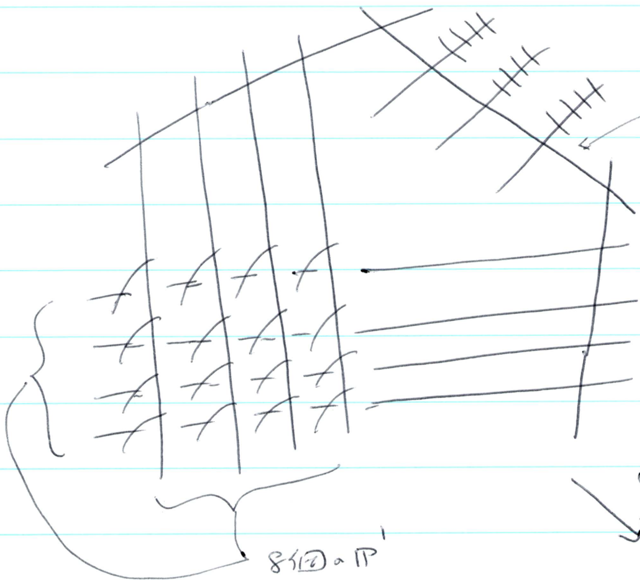


$\mathbb{P}^2 \xleftarrow{2:1} \overline{Y}$ 21点上で node.

$w^2 = x_0x_1x_2(x_0^3+x_1^3+x_2^3)$.

$\nwarrow 2:1$

$\tilde{Y}$ $\nwarrow$ mini. resolution



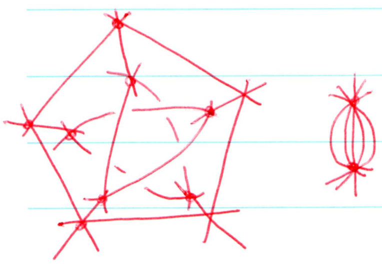
contract $12\ \mathbb{P}^1$

$\overline{X}$ = can. cover of X

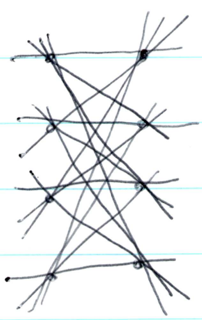
$8 A_1$-sig, D_4-sig.

VII

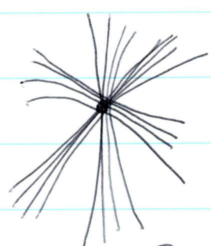
μ_2 derivation (= vector field)



15本の$\mathbb{P}^1$

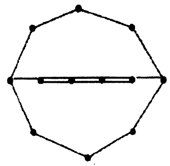
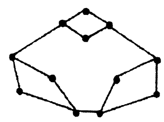
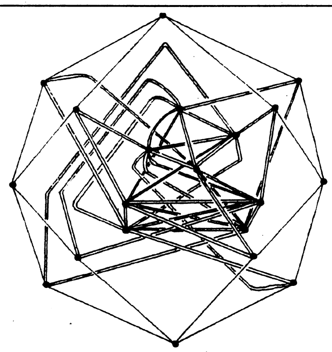
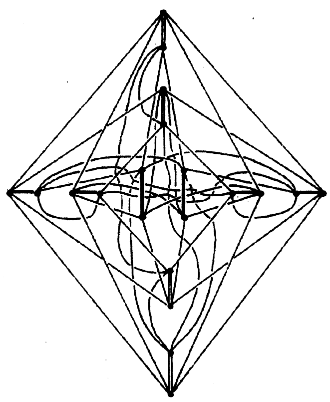


16個の$\mathbb{P}^1$



12個

mod 2 でこの図

Type	Dual Graph of (-2)-curves	Aut	Aut_{nt}	$\mathrm{char}(k)$	Moduli
I		D_4	$\mathbb{Z}/2\mathbb{Z}$	*any*	$\mathbb{A}^1 - \{0, -256\}$
II		$\mathfrak{S}_4$	$\{1\}$	*any*	$\mathbb{A}^1 - \{0, -64\}$
III		$(\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2) \rtimes D_4$	$\mathbb{Z}/2\mathbb{Z}$	$\neq 2$	*unique*
IV		$(\mathbb{Z}/2\mathbb{Z})^4 \rtimes (\mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z})$	$\{1\}$	$\neq 2$	*unique*

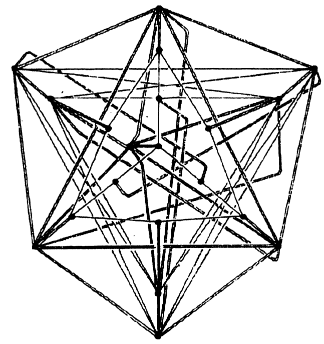
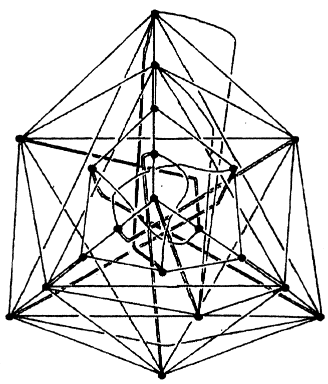
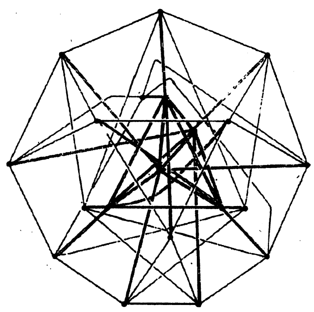
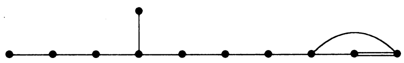
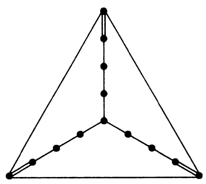
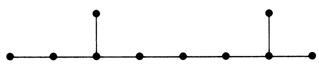
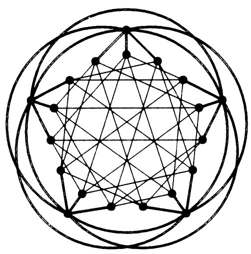
V		$\mathfrak{S}_4 \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\neq 2, 3$	*unique*
VI		$\mathfrak{S}_5$	$\{1\}$	$\neq 3, 5$	*unique*
VII		$\mathfrak{S}_5$	$\{1\}$	$\neq 2, 5$	*unique*

TABLE 1. Classification

(A) Classification

Type	Dual Graph of (-2)-curves
$\tilde{E}_8$	
$\tilde{E}_7+\tilde{A}_1{}^{(1)}$	
$\tilde{E}_6+\tilde{A}_2$	
$\tilde{D}_8$	
VII	

(B) Examples

$\mathrm{Aut}(X)$	$\mathrm{Aut}_{ct}(X)$	dim
$\mathbf{Z}/11\mathbf{Z}$	$\mathbf{Z}/11\mathbf{Z}$	0
$\mathbf{Z}/2\mathbf{Z}$ or $\mathbf{Z}/14\mathbf{Z}$	$\{1\}$ or $\mathbf{Z}/7\mathbf{Z}$	1 or 0
$\mathbf{Z}/5\mathbf{Z}\times\mathfrak{S}_3$	$\mathbf{Z}/5\mathbf{Z}$	0
Q_8	Q_8	1
$\mathfrak{S}_5$	$\{1\}$	0

TABLE 1

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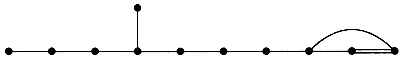
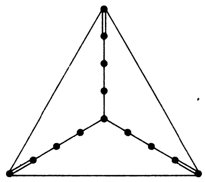
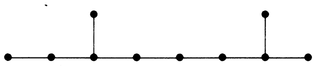
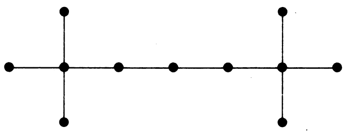
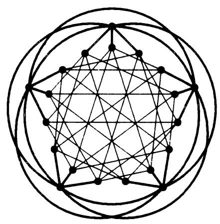
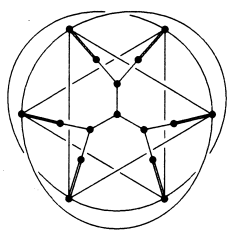
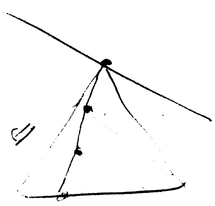
(A) Classification

Type	Dual Graph of (-2)-curves
$\tilde{E}_8$	
$\tilde{E}_7+\tilde{A}_1^{(1)}$	
$\tilde{E}_7+\tilde{A}_1^{(2)}$	
$\tilde{E}_6+\tilde{A}_2$	
$\tilde{D}_8$	
$\tilde{D}_4+\tilde{D}_4$	
VII	
VIII	

(B) Examples

$\mathrm{Aut}(X)$	$\mathrm{Aut}_{nt}(X)$	dim
$\{1\}$	$\{1\}$	1
$\mathbf{Z}/2\mathbf{Z}$	$\{1\}$	2
$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	1
$\mathfrak{S}_3$	$\{1\}$	1
$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	2
$(\mathbf{Z}/2\mathbf{Z})^3$	$(\mathbf{Z}/2\mathbf{Z})^2$	2
$\mathfrak{S}_5$	$\{1\}$	1
$\mathfrak{S}_4$	$\{1\}$	1

TABLE 2



REFERENCES

[1] W. Barth and C. Peters, *Automorphisms of Enriques surfaces*, Invent. Math., **73** (1983), 383–411.

[2] E. Bombieri and D. Mumford, *Enriques' classification of surfaces in char. p, III*, Invent. Math., **35** (1976), 197–232.

[3] R. Borcherds, *The moduli space of Enriques surfaces and the fake monster Lie superalgebra*, Topology, **35** (1996), 699–710.

[4] J.H. Conway, *The automorphism group of the* 26 *dimensional even Lorentzian lattice*, J. Algebra **80** (1983), 159-163.

[5] I. Dolgachev, *On automorphisms of Enriques surfaces,* Invent. Math., **76** (1984), 163–177.

[6] T. Ekedahl and N. I. Shepherd-Barron, *On exceptional Enriques surfaces,* arXiv:math/0405510.

[7] G. Fano, *Superficie algebriche di genere zero e bigenere uno e loro casi particolari*, Rend. Circ. Mat. Palermo, **29** (1910), 98–118.

[8] E. Horikawa, *On the periods of Enriques surfaces* I, II, Math. Ann., **234** (1978), 78–108; ibid **235** (1978), 217–246.

[9] T. Katsura and S. Kondō, *On Enriques surfaces in characteristic* 2 *with a finite group of automorphisms,* J. Algebraic Geometry **27** (2018), 173–202.

[10] T. Katsura, S. Kondō, G. Martin, *Classification of Enriques surfaces with finite automorphism group in characteristic* 2, arXiv:1703.09609v2.

[11] S. Kondō, *Enriques surfaces with finite automorphism groups*, Japanese J. Math., **12** (1986), 191–282.

[12] S. Kondō, *The rationality of the moduli space of Enriques surfaces*, Compositio Math., **91** (1994), 159–173.

[13] S. Kondō, *Classification of Enriques surfaces covered by the supersingular $K3$ surface with Artin invariant* 1 *in characteristic* 2, arXiv:1812.02020.

[14] C. Liedtke, *Arithmetic moduli and liftings of Enriques surfaces,* J. Reine Angew. Math., **706** (2015), 35–65.

[15] G. Martin, *Enriques surfaces with finite automorphism group in positive characteristic*, to appear in Algebraic Geometry; arXiv:1703.08419.

[16] V. Nikulin, *On a description of the automorphism groups of Enriques surfaces,* Soviet Math. Dokl., **30** (1984), 282–285.

[17] E. B. Vinberg, *Some arithmetic discrete groups in Lobachevskii spaces*, in "Discrete subgroups of Lie groups and applications to Moduli", Tata-Oxford (1975), 323–348.