Dispersionless Integrable Systems and Löwner type Equations

Takashi Takebe
Faculty of Mathematics/ International Laboratory
of Representation Theory and Mathematical Physics,
National Research University Higher School of Economics,
Moscow, Russia

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0 Introduction

- Integrable hierarchies = 'solvable' systems with infinitely many variables (e.g., $t = (t_1, t_2, t_3, \dots)$).
- Dispersionless integrable hierarchies

  = quasi-classical limits of certain integrable hierarchies.
- One-variable reduction: (resp. N-variable reduction):

  solutions depend on ∞-many variables only through one function, e.g., $\lambda(t)$. (resp. N functions, $\lambda_i(t)$, $i = 1, \dots, N$).

Today's topic

One-variable reduction of the dispersionless KP
(resp. Toda, BKP, DKP) hierarchy

$$\Updownarrow$$

The chordal (resp. radial, quadrant, annulus) Löwner equation.

For N-variable reduction: Löwner equations + compatibility conditions.

Plan of the talk:

1. Brief introduction to integrable systems.
2. KP hierarchy and Toda lattice hierarchy.
3. Dispersionless hierarchies.
4. Dispersionless Hirota equations.
5. dKP hierarchy and chordal Löwner equation.
6. Other examples.

Disclaimer: In this talk everything is quite "algebraic":

- "functions" = formal power series
- "operators" = elements of non-commutative rings

Only algebraic structure is studied.
(& "genericity conditions" often omitted, ...)

1 What are "integrable systems"?

For systems with *finite* degrees of freedom,
$\exists$ well established/defined geometric criteria of integrability.

- Frobenius integrability condition
- Liouville integrability condition (for Hamiltonian systems)
  = "existence of sufficiently many conserved quantities"

Examples: Kepler motion, Tops (Euler, Lagrange, Kowalevski)

How about "integrable systems" with infinite degrees of freedom?
No definite consensus. Let us review the history and list examples.

Modern theory of integrable systems began with the discovery of *remarkable solutions of non-linear partial differential equations* = "SOLITONS" in 1960's.

Soliton = particle-like stable solitary wave

(Numerical experiments by Zabusky and Kruskal (1965).)

Examples of soliton equations:

- KdV equation (1895): $u = u(x,t)$, $u_t - 3uu_x - \frac{1}{4}u_{xxx} = 0$.
- KP equation (1970): $u = u(x,y,t)$,
$$\frac{3}{4}u_{yy} - (u_t - 3uu_x - \frac{1}{4}u_{xxx})_x = 0.$$
- Sine-Gordon equation : $u = u(x,t)$, $u_{tt} - u_{xx} - \sin u = 0$.
- Toda lattice (1967): $u_n = u_n(t)$, $u_{n,tt} = e^{u_{n-1}-u_n} - e^{u_n - u_{n+1}}$.

Surprisingly, such soliton equations are *solvable* in spite of its nonlinearity!

- inverse scattering method, Lax pairs
- algebro-geometric solutions
- Hirota's bilinear method

$\Longrightarrow$ various generalisations of soliton equations were found.

Why are they solvable? $\Longrightarrow$ discovery of

- infinitely many conserved quantities/ symmetries
- moduli space of solutions (e.g., ∞-dimensional Grassmann manifold for KP hierarchy)

$\Longrightarrow$ relation to algebra
(e.g., representation theory of ∞-dimensional Lie algebras).

Let us examine the KP and the Toda lattice hierarchies as examples.

2 KP hierarchy and Toda lattice hierarchy

KP hierarchy: integrable nonlinear system of PDE.

- $u_i(t)$ $(i = 2, 3, \dots)$: unknown functions
- $t = (t_1, t_2, t_3, \dots)$: independent variables
  $(x = t_1,\ \partial = \partial/\partial x.)$

The *Lax operator*: $L = \partial + u_2(t)\partial^{-1} + u_3(t)\partial^{-2} + \cdots$.
— "generating operator" of u_i's.

Here,

- $f(x)\partial^m$ $(m \in \mathbb{Z})$: microdifferential operators. Multiplication defined by

$$\big(f(x)\partial^m\big)\big(g(x)\partial^n\big) = \sum_{r=0}^{\infty} \binom{m}{r} f g^{(r)} \partial^{m+n-r},$$

where

$$\binom{m}{r} = \frac{m(m-1)\cdots(m-r+1)}{r!}.$$

(Recall $m \in \mathbb{Z}$.)

$\mathcal{E}$ = algebra of microdiff. operators $\supset \mathcal{D}$ = algebra of diff. operators

KP hierarchy: (Lax representation)

(KP) $$\frac{\partial L}{\partial t_n} = [B_n, L] \qquad (n = 1, 2, \dots; B_n = (L^n)_{\geq 0}).$$

Notations: $P = \sum_{n\in\mathbb{Z}} a_n \partial^n \ \to P_{\geq 0} := \sum_{n \geq 0} a_n \partial^n$, $P_{<0} := \sum_{n<0} a_n \partial^n$.

This includes the KP equation for $u = u_2$:

$$\tfrac{3}{4}u_{t_2t_2} - \left(u_{t_3} - 3uu_x - \tfrac{1}{4}u_{xxx}\right)_x = 0$$

$\because$) First two equations $\dfrac{\partial L}{\partial t_2} = [B_2, L]$ and $\dfrac{\partial L}{\partial t_3} = [B_3, L]$ are expanded as

$$\frac{\partial u_2}{\partial t_2}\partial^{-1} + \frac{\partial u_3}{\partial t_2}\partial^{-2} + \cdots = (u_2'' + 2u_3')\partial^{-1} + (u_3'' + 2u_4' + 2u_2u_2')\partial^{-2} + \cdots .$$

$$\frac{\partial u_2}{\partial t_3}\partial^{-1} + \frac{\partial u_3}{\partial t_3}\partial^{-2} + \cdots = (3u_3'' + 3u_4' + 6u_2u_2' + u_2''')\partial^{-1} + \cdots .$$

($(\cdot)' = \partial(\cdot)/\partial x$.)

Comparing the coefficients of ∂^{-1} and ∂^{-2} we have

$$\frac{\partial u_2}{\partial t_2} = u_2'' + 2u_3', \qquad \frac{\partial u_3}{\partial t_2} = u_3'' + 2u_4' + 2u_2u_2',$$

$$\frac{\partial u_2}{\partial t_3} = 3u_3'' + 3u_4' + 6u_2u_2' + u_2''',$$

Eliminating u_3 and u_4 we obtain the KP equation. □

- KP hierarchy

  = set of compatibility conditions for the linear problem for $\Psi = \Psi(t; z)$:

$$L\Psi = z\Psi, \qquad \frac{\partial \Psi}{\partial t_n} = B_n\Psi.$$

  (z: spectral parameter)

- $\exists\ \Psi = (1 + w_1(t)z^{-1} + w_2(t)z^{-2} + \cdots)e^{\sum t_n z^n}$

  or, $W := 1 + w_1(t)\partial^{-1} + w_2(t)\partial^{-2} + \cdots$, $\Psi = We^{\sum t_n z^n}$.

- $L = W\partial W^{-1}$, $\dfrac{\partial W}{\partial t_n} = -(L^n)_{<0}W$.

- L satisfies (KP) $\Leftrightarrow \exists\ \tau(t)$ (tau function) such that
  - Ψ is expressed by τ as $\Psi(t;z) = \dfrac{\tau(t-[z^{-1}])}{\tau(t)} e^{\sum t_n z^n}$,
    $t = (t_n)_{n=1,2,\ldots}$, $t - [z^{-1}] = \left(t_n - \frac{z^{-n}}{n}\right)_{n=1,2,\ldots}$.
  - $\tau(t)$ satisfies a series of bilinear differential equations (the Hirota equations).
    The generating function of the Hirota equations:

$$
\begin{aligned}
&\alpha_1(\alpha_3-\alpha_2)\tau(t+[\alpha_1])\,\tau(t+[\alpha_2]+[\alpha_3]) \\
&-\alpha_2(\alpha_3-\alpha_1)\tau(t+[\alpha_2])\,\tau(t+[\alpha_1]+[\alpha_3]) \\
&+\alpha_3(\alpha_2-\alpha_1)\tau(t+[\alpha_3])\,\tau(t+[\alpha_1]+[\alpha_2]) = 0,
\end{aligned}
$$

- Solutions of the KP hierarchy are parametrised by the Sato Grassmann manifold (an ∞-dimensional Grassmann manifold).
  $\exists \mathcal{V}$: $\mathcal{E}$-module $\supset V^{\varnothing}$: $\mathcal{D}$-module.

$$\text{Sato Grassmann manifold} \ni W^{-1}|_{t=0} V^{\varnothing}.$$

- τ-function & its derivatives = Plücker coordinates.
- Hirota equations = defining equations of the Grassmann manifold (Plücker relations).
- ∞-dimensional symmetry:
  $GL(\infty)$ acts on the Sato Grassmann manifold $= GL(\infty)/P_{\infty/2}$.

  (cf. finite dimensional Grassmann manifold $= GL(N)/P$,

$$P = \left\{ \left(\begin{array}{ccc|ccc} * & \cdots & * & & & \\ \vdots & \ddots & \vdots & & * & \\ * & \cdots & * & & & \\ \hline & & & * & \cdots & * \\ & 0 & & \vdots & \ddots & \vdots \\ & & & * & \cdots & * \end{array} \right) \right\}.)$$

Variants:

- (KP) + constraint $L^2 = \partial^2 + 2u$

  $\Longrightarrow$ KdV hierarchy, which contains the KdV equation for u.

  This has the symmetry of $sl(2, \mathbb{C}[t, t^{-1}])\oplus$ (central extension),

  i.e., $A_1^{(1)}$-type affine Lie algebra.

- (KP) + constraint $L^* = -\partial L \partial^{-1}$

  (Notation: $(a(x)\partial^n)^* := (-\partial)^n a(x)$ is the formal adjoint operator.)

  $\Longrightarrow$ BKP hierarchy, which has the symmetry of $so(2\infty + 1)$ (B_∞-type).

- There are CKP and DKP hierarchies corresponding to C_∞ and D_∞ type symmmetries, but the definitions are involved.

  (Usually defined by the Hirota bilinear equations.)

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Toda lattice hierarchy: $\phi, u_n, \bar{u}_n$: unknown functions of s, $t = (t_n)_{n\in\mathbb{Z}, n\neq 0}$.

$$
\begin{aligned}
L &= e^{\phi} e^{\partial_s} + u_1 + u_2 e^{-\partial_s} + u_3 e^{-2\partial_s} + \cdots , \\
\bar{L}^{-1} &= e^{\phi} e^{-\partial_s} + \bar{u}_1 + \bar{u}_2 e^{\partial_s} + \bar{u}_3 e^{2\partial_s} + \cdots , \\
B_n &= \begin{cases} (L^n)_{>0} + \frac{1}{2}(L^n)_0, & (n > 0), \\ (\bar{L}^{-n})_{<0} + \frac{1}{2}(\bar{L}^{-n})_0, & (n < 0). \end{cases}
\end{aligned}
$$

Notations:

- $e^{n\partial_s} f(s) = f(s+n)$: difference operator.
- $A = \sum_{n\in\mathbb{Z}} a_n e^{n\partial_s} \to A_S = \sum_{n\in S} a_n e^{n\partial_s}$ for $S =$ "> 0", "< 0" and "0".

Toda lattice hierarchy: (Lax representation)

(Toda) $$\frac{\partial L}{\partial t_n} = [B_n, L], \qquad \frac{\partial \bar{L}}{\partial t_n} = [B_n, \bar{L}], \qquad (n \in \mathbb{Z},\ n \neq 0).$$

- Parametrisations of solutions, τ function etc. are known.
- $n = \pm 1 \Longrightarrow$ the 2d Toda equation:
  $$\frac{\partial^2}{\partial t_1\, \partial t_{-1}} \phi(s,t) = e^{\phi(s-1,t)-\phi(s,t)} - e^{\phi(s,t)-\phi(s+1,t)}.$$
- 2d Toda eq. + constraint $\phi(s+2,t) = \phi(s,t)$
  (+ change of variables) $\Longrightarrow$ Sine-Gordon eq.
- (Toda) + constraint: $L = \bar{L}^{-1}$
  $\Longrightarrow$ 1d Toda hierarchy (which contains the Toda lattice for ϕ).

3 Dispersionless hierarchies

- ∂, e^{∂_s} $\to$ commutative symbols.
- commutator $[,]$ $\to$ Poisson bracket $\{,\}$.

$\Longrightarrow$ dispersionless KP/Toda lattice hierarchies.

- dispersionless KP hierarchy: $\partial^n \to w^n$, $\{w, x\} = 1$.

$$\mathcal{L} = w + u_2(t)w^{-1} + u_3(t)w^{-2} + \cdots, \qquad \mathcal{B}_n = (\mathcal{L}^n)_{\geq 0}.$$

($\mathcal{P} = \sum_{n\in\mathbb{Z}} a_n w^n \to \mathcal{P}_S := \sum_{n\in S} a_n w^n$ for $S =$ "≥ 0", "< 0" etc.)

dKP hierarchy

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\} \qquad (n = 1, 2, \dots).$$

Why "*dispersionless*"?

KP hierarchy $\ni$ KdV eq.: $u_t - 3uu_x - \overbrace{\tfrac{1}{4}u_{xxx}}^{\text{dispersion term}} = 0.$

dKP hierarchy $\ni$ dispersionless KdV eq.: $u_t - 3uu_x = 0.$

- dispersionless Toda lattice hierarchy: $e^{n\partial_s} \to w^n$, $\{w, s\} = w$.

$$\mathcal{L} = e^{\phi}w + u_1 + u_2 w^{-1} + \cdots, \quad \tilde{\mathcal{L}}^{-1} = e^{\phi}w^{-1} + \bar{u}_1 + \bar{u}_2 w + \cdots,$$

$$\mathcal{B}_n = \begin{cases} (\mathcal{L}^n)_{>0} + \frac{1}{2}(\mathcal{L}^n)_0, & (n > 0), \\ (\tilde{\mathcal{L}}^{-n})_{<0} + \frac{1}{2}(\tilde{\mathcal{L}}^{-n})_0, & (n < 0). \end{cases}$$

dToda hierarchy

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \quad \frac{\partial \tilde{\mathcal{L}}}{\partial t_n} = \{\mathcal{B}_n, \tilde{\mathcal{L}}\}, \quad (n \in \mathbb{Z}, n \neq 0).$$

For dKP/dToda hierarchies, ∞-dimensional symmetries (w_∞-algebra), parametrisation of solutions ($\longleftrightarrow$ canonical transformations) are known. ([Takasaki-T.] 1991–1995)

4 Dispersionless Hirota equations

(Maybe you feel flavour of complex analysis...)

[Takasaki-T. (1995)] τ of KP (with $\hbar$) $= \exp\bigl(\hbar^{-2}\mathcal{F} + O(\hbar^{-1})\bigr)$.
dHirota$= \lim_{\hbar\to 0}$ Hirota eq.

Teo's formulation (2002)

$\mathcal{L}(t;w) = w + u_1(t)w^{-1} + u_2(t)w^{-2} + \cdots$.
$k(t;z)$: inverse fuction of $\mathcal{L}(t;w)$ with respect to w:
$\mathcal{L}(t;k(t;z)) = z$, $\qquad k(t;\mathcal{L}(t;w)) = w$.
Grunsky coefficients $b_{mn}(t)$ of $k(t;z)$ (... for the Bieberbach conjecture):

(dH1) $$\log\frac{k(t;z_1)-k(t;z_2)}{z_1-z_2} = -\sum_{m,n=1}^{\infty} b_{mn}(t) z_1^{-m} z_2^{-n}.$$

$$\iff \quad \mathcal{L}^n + \sum_{m=1}^{\infty} n b_{nm}(t)\mathcal{L}^{-m} = (\text{polynomial in } w) = (\mathcal{L}^n)_{\geq 0}.$$

In particular

(dH2) $$k(t;z) = z + \sum_{m=1}^{\infty} b_{1,m} z^{-m}.$$

Theorem

$\mathcal{L}(t;w)$: solution of dKP
$\iff$ There exists $\mathcal{F}(t)$ such that $\dfrac{\partial^2\mathcal{F}}{\partial t_m \partial t_n} = -mnb_{mn}(t)$.

(dH1&2) rewritten in terms of $\mathcal{F}(t)$:

dispersionless Hirota eq.

(dH) $$e^{D(z_1)\,D(z_2)\mathcal{F}} = -\frac{\partial_1\bigl(D(z_1)-D(z_2)\bigr)\mathcal{F}}{z_1-z_2}.$$

Notation: $D(z) := \sum \dfrac{z^{-n}}{n}\dfrac{\partial}{\partial t_n}$.

($\exists$ similar theorem for dToda.)

5 Dispersionless KP and Löwner equation

Unexpected relation of the (chordal) Löwner equation and the dispersionless KP hierarchy was found by

- Gibbons-Tsarev (1999) for t_1 and t_2; Yu-Gibbons (2000) in general (direct computation).
- Mañas-Martínez Alonso-Medina (2002): proof by "S function" $\fallingdotseq \log \Psi$.
- T.-Teo-Zabrodin (2006): proof by dHirota eq.

Chordal Löwner equation:
$H = \{\operatorname{Im} z > 0\}$: the upper half plane. $\Gamma : [a,b] \to H$: Jordan curve.
$g(\lambda; z) : H \setminus \Gamma([a,\lambda]) \xrightarrow{\sim} H$: conformal mapping normalised as

$$g(\lambda; z) = z + a_1(\lambda) z^{-1} + O(z^{-2}) \quad (z \to \infty), \qquad g(0; z) = z.$$

$\Longrightarrow \exists\, U(\lambda)$ s.t.

Chordal Löwner equation

$$\frac{\partial g}{\partial \lambda}(\lambda; z) = \frac{1}{g(\lambda; z) - U(\lambda)} \frac{d a_1}{d\lambda}.$$

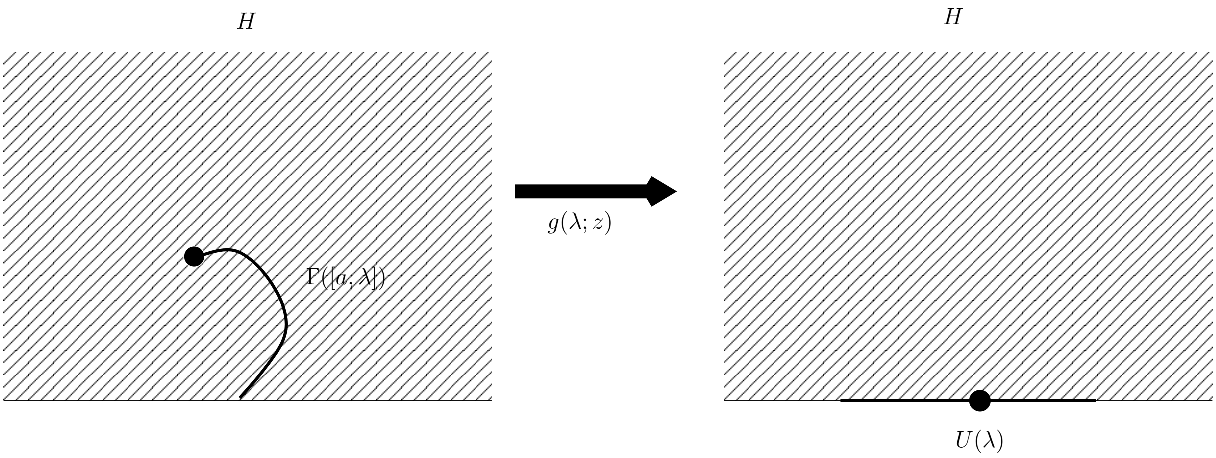

Figure 1: The slit mapping between $H \setminus \Gamma([a,\lambda])$ and H.

One variable reduction of dKP

Theorem

$\mathcal{L}(t;w)$ is a solution of dKP such that:
$\exists$ functions $\lambda(t)$ & $f(\lambda,w)$: $\mathcal{L}(t;w) = f(\lambda(t),w)$.
$\Longrightarrow$
(i) $f(\lambda,w)$ is the inverse function of a solution $g(\lambda,z)$ of the chordal Löwner eq. ($f(\lambda,g(\lambda,z)) = z, \quad g(\lambda,f(\lambda,w)) = w$.)
(ii) $\lambda(t)$ satisfies $\dfrac{\partial\lambda}{\partial t_n} = \dfrac{\partial\Phi_n}{\partial w}(\lambda;U(\lambda))\dfrac{\partial\lambda}{\partial t_1}$ $(n = 1,2,\dots)$

Here, $\Phi_n(\lambda;w) = (f(\lambda,w)^n)_{\geq 0}$: Faber polynomial of g.
(Polynomial part of $f(\lambda,w)^n$ w.r.t. w.)

Conversely:

Theorem

$g(\lambda,z)$: solution of the chordal Löwner equation.
$f(\lambda,w) = w + O(w^{-1})$: inverse function of g,
i.e., $f(\lambda,g(\lambda,z)) = z, \quad g(\lambda,f(\lambda,w)) = w$.
$\lambda(t)$: solution of $\dfrac{\partial\lambda}{\partial t_n} = \dfrac{\partial\Phi_n}{\partial w}(\lambda;U(\lambda))\dfrac{\partial\lambda}{\partial t_1}$ $(n = 1,2,\dots)$
$\Longrightarrow \mathcal{L}(t,w) := f(\lambda(t),w)$ is a solution of dKP.

Remark: The equation for $\lambda(t)$ is solved implicitly by the relation

$$t_1 + \sum_{n=2}^{\infty} t_n \frac{\partial\Phi_n}{\partial w}(\lambda;U(\lambda)) = R(\lambda).$$

$R(\lambda)$: arbitrary generic function. (Tsarev's generalised hodograph method.)

Idea of the proof ([TTZ]):
Enough to show the existence of $\mathcal{F}$, i.e., the integrability condition of

$$\frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_n} = -mnb_{mn}(t).$$

($b_{mn}(t)$: the Grunsky coefficients of $g(\lambda; z)$.)

$\iff mn\frac{\partial b_{mn}}{\partial t_k} = kn\frac{\partial b_{kn}}{\partial t_m}$. (Note: $b_{mn} = b_{nm}$.)

$\iff \frac{1}{k}\frac{\partial b_{mn}}{\partial t_k}$ is symmetric in (k, m, n).

$\iff c(z_1, z_2, z_3) := -\sum_{k,m,n=1}^{\infty} z_1^{-k} z_2^{-m} z_3^{-n} \frac{1}{k}\frac{\partial b_{mn}}{\partial t_k}$ is symmetric in (z_1, z_2, z_3).

By the definition of the Grunsky coefficients,

$$c(z_1, z_2, z_3) = \frac{D(z_1)\big(g(t; z_2) - g(t; z_3)\big)}{g(t; z_2) - g(t; z_3)}, \quad g(t; z) := g(\lambda(t); z).$$

The generating function of $\Phi(w)$:

$$\log \frac{g(t; z) - w}{z - \zeta} = -\sum_{n=1}^{\infty} \Phi_n(w) \frac{z^{-n}}{n} + \sum_{n=1}^{\infty} \zeta^n \frac{z^{-n}}{n}.$$

By differentiating by w and substituting $w = U(\lambda(t))$,

$$\frac{1}{g(t; z) - U(\lambda(t))} = \sum_{n=1}^{\infty} \Phi_n'(U(\lambda(t))) \frac{z^{-n}}{n},$$

The chordal Löwner equation + the equation for $\lambda(t) \Longrightarrow$

$$c(z_1, z_2, z_3) = -\frac{\partial_1 \lambda}{(g(t; z_1) - U(t))(g(t; z_2) - U())(g(t; z_3) - U(t))} \frac{\partial a_1}{\partial \lambda}.$$

($U(t) = U(\lambda(t))$.) □

<u>N-variable version:</u>
$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$.
$g = g(\boldsymbol{\lambda}; z) = z + a_1(\boldsymbol{\lambda}) z^{-1} + O(z^{-2})$, $f = f(\boldsymbol{\lambda}; w)$: inverse function of g.

Theorem

Suppose that $g(\boldsymbol{\lambda}; z)$ is a solution of

$$\frac{\partial g}{\partial \lambda_i}(\boldsymbol{\lambda}; z) = \frac{1}{g(\boldsymbol{\lambda}; z) - U_i(\boldsymbol{\lambda})} \frac{\partial a_1}{\partial \lambda_i}$$

for all $i = 1, \ldots, N$ and that each $\lambda_i(t)$ satisfies

$$\frac{\partial \lambda_i}{\partial t_n} = \frac{\partial \Phi_n}{\partial w}(\boldsymbol{\lambda}; U_i(\boldsymbol{\lambda})) \frac{\partial \lambda_i}{\partial t_1}.$$

$\Longrightarrow \mathcal{L}(t; w) := f(\lambda(t); w)$ is a solution of dKP.

Functions $a_i(\boldsymbol{\lambda})$ and $\{U_i(\boldsymbol{\lambda})\}$ should satisfy compatibility conditions.
Gibbons-Tsarev system:

$$\frac{\partial^2 a_1}{\partial \lambda_i \, \partial \lambda_j} = \frac{-2}{(U_i - U_j)^2} \frac{\partial a_1}{\partial \lambda_i} \frac{\partial a_1}{\partial \lambda_j},$$
$$\frac{\partial U_j}{\partial \lambda_i} = \frac{1}{U_i - U_j} \frac{\partial a_1}{\partial \lambda_i}.$$

6 Other examples

- mKP hierarchy $\longleftrightarrow$ chordal Löwner-like equation
  (Mañas-Martínez Alonso-Medina)
- Toda hierarchy $\longleftrightarrow$ radial Löwner equation (T.-Teo-Zabrodin, ...)
- BKP hierarchy $\longleftrightarrow$ quadrant Löwner equation (T.)
- DKP hierarchy $\longleftrightarrow$ annulus Löwner (Goluzin-Komatu) equation
  (Akhmedova-Zabrodin(-T.))

dBKP hierarchy: dKP + constraint: $\mathcal{L}(w) = -\mathcal{L}(-w)$.
Quadrant Löwner equation:

$$\frac{\partial g}{\partial \lambda} = \frac{g}{V^2 - g^2} \frac{du}{d\lambda}.$$

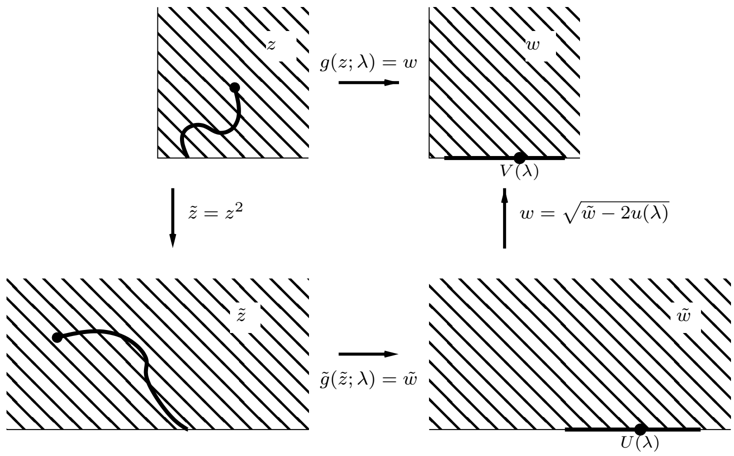

Figure 2: Conformal mapping from a slit domain to the quadrant.

Problem

WHY do Löwner type equations give solutions of dispersionless integrable hierarchies?

Thank you for your attention.

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