## "K3 Surface and Mathieu Moonshine"

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♣ Elliptic genus of K3

Elliptic genus in string theory is expressed as

$$Z_{elliptic}(z;\tau) = Tr_{\mathcal{H}_L \times \mathcal{H}_R}(-1)^{F_L+F_R} e^{4\pi i z J^3_{L,0}} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{c}{24}}$$

and describes the topological invariants of the target manifold and counts the number of BPS states in the theory. Here $L_0$ denotes the zero mode of

the Visasoro operators and $F_L$ and $F_R$ are left and right moving fermion numbers. In ellitpic genus the right moving sector is frozen to the supersymmetric ground states (BPS states) while in the left moving sector all the states in the Hilbert space $\mathcal{H}_L$ contribute.

SUSY algebra

$$\{\bar{G}^i_0, \bar{G}^{*\,j}_0\} = 2\delta^{ij}\bar{L}_0 - \frac{k}{2}\delta^{ij}, \ (i,j=1,2) \Longrightarrow \bar{L}_0 \geq \frac{k}{4}$$

BPS states

$$\bar{L}_0 = \frac{k}{4}$$

Elliptic genus of K3 surface is known: EOTY

$$Z_{K3}(z;\tau) = 8\left[\left(\frac{\theta_2(z;\tau)}{\theta_2(0;\tau)}\right)^2 + \left(\frac{\theta_3(z;\tau)}{\theta_3(0;\tau)}\right)^2 + \left(\frac{\theta_4(z;\tau)}{\theta_4(0;\tau)}\right)^2\right]$$

$$Z_{K3}(z=0) = 24, \quad Z_{K3}(z=\frac{1}{2}) = 16 + O(q),$$
$$Z_{K3}(z=\frac{1+\tau}{2}) = 2q^{-\frac{1}{2}} + O(q^{\frac{1}{2}})$$

Elliptic genus of a complex D-dimensional manifold is a Jacobi form of weight=0 and index=D/2. When D=2, space of Jacobi form is one-dimensional and given by the above formula.

Jacobi form (weight $k$ and index $m$)

$$\varphi(\tau, z + a\tau + b) = e^{-2\pi i m(a^2\tau + 2az)}\varphi(\tau, z)$$
$$\varphi(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}) = (c\tau + d)^k e^{\frac{2\pi i m c z^2}{c\tau + d}}\varphi(\tau, z)$$

String theory on K3 has an N=4 superconformal symmetry and its states fall into representations of N=4 superconformal algebra (SCA). N=4 SCA contains an affine $SU(2)_k$ symmetry and has a central charge $c = 6k$. $k = n$ case describes complex-$2n$ dimensional hyperKähler manifolds.

We would like to study the decomposition of the elliptic genus in terms of irreducible characters of N=4 SCA. In N=4 SCA, hightest-weight states $|h, \ell\rangle$ are charactered by

$$L_0|h,\ell\rangle = h|h,\ell\rangle, \qquad J_0^3|h,\ell\rangle = \ell|h,\ell\rangle$$

and the theory possesses two different type of representations, BPS and non-BPS representations. In

the case of $k = 1$ there are representations (in Ramond sector)

| | | |
|---|---|---|
| BPS rep. | $h = \frac{1}{4};$ | $\ell = 0, \frac{1}{2}$ |
| non-BPS rep. | $h > \frac{1}{4};$ | $\ell = \frac{1}{2}$ |

Character of a representation is given by

$$Tr_{\mathcal{R}}(-1)^F q^{L_0} e^{4\pi i z J_0^3}$$

Its index is given by the value at $z = 0$, $Tr_{\mathcal{R}}(-1)^F q^{L_0}$. BPS representations have a non-vanishing index

$$\text{index (BPS, } \ell = 0) = 1$$
$$\text{index (BPS, } \ell = \tfrac{1}{2}) = -2$$

Character function of $\ell = 0$ BPS representation has the form

$$ch^{\tilde{R}}_{h=\frac{1}{4},\ell=0}(z;\tau) = \frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}\mu(z;\tau)$$

where

$$\mu(z;\tau) = \frac{-ie^{\pi iz}}{\theta_1(z;\tau)}\sum_n (-1)^n \frac{q^{\frac{1}{2}n(n+1)}e^{2\pi inz}}{1-q^n e^{2\pi iz}}$$

On the other hand the character of non-BPS representations are given by

$$ch^{\tilde{R}}_{h>\frac{1}{4},\ell=\frac{1}{2}} = q^{h-\frac{3}{8}}\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}$$

These have vanishing index

$$\text{index (non-BPS rep)} = 0$$

At the unitarity bound non-BPS representation splits into a sum of BPS representations

$$\lim_{h\to\frac{1}{4}} q^{h-\frac{3}{8}}\frac{\theta_1^2}{\eta^3} = ch^{\tilde{R}}_{h=\frac{1}{4},\ell=\frac{1}{2}} + 2ch^{\tilde{R}}_{h=\frac{1}{4},\ell=0}$$

Function $\mu(z;\tau)$ is a typical example of the so-called Mock theta functions (Lerch sum or Appell function). Mock theta functions look like theta functions but they have anomalous modular transformation laws and are difficult to handle. Recently there have been developments in understanding the nature of Mock theta functions initiated by Zwegers who has developed a way to improve their modular properties. We will adopt his method of handling Mock theta functions.

It is possible to derive the following idenities

$$\begin{aligned} ch^{\tilde{R}}_{h=\frac{1}{4},\ell=0}(z;\tau) &= \left(\frac{\theta_2(z;\tau)}{\theta_2(0;\tau)}\right)^2 + \mu_2(\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3} \\ &= \left(\frac{\theta_3(z;\tau)}{\theta_3(0;\tau)}\right)^2 + \mu_3(\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3} \\ &= \left(\frac{\theta_4(z;\tau)}{\theta_4(0;\tau)}\right)^2 + \mu_4(\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3} \end{aligned}$$

where

$$\mu_2(\tau)=\mu(z=\frac{1}{2};\tau),\ \mu_3(\tau)=\mu(z=\frac{1+\tau}{2};\tau),\ \mu_4(\tau)=\mu(z=\frac{\tau}{2};\tau)$$

$$\mu(z;\tau) = \frac{-ie^{\pi iz}}{\theta_1(z;\tau)} \sum_n (-1)^n \frac{q^{\frac{1}{2}n(n+1)} e^{2\pi inz}}{1 - q^n e^{2\pi iz}}$$

Then we can rewrite the ellitpic genus as

$$Z_{K3} = 24 ch^{\tilde{R}}_{h=\frac{1}{4},\ell=0}(z;\tau) - 8 \sum_{i=2}^{4} \mu_i(\tau) \frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}$$

Using q-expansion of functions $\mu_i$ we find

$$8\left(\mu_2(\tau) + \mu_3(\tau) + \mu_4(\tau)\right) = -2 \sum_{n=0} A(n) q^{n-\frac{1}{8}}$$

$$Z_{K3} = 24 ch^{\tilde{R}}_{h=\frac{1}{4},\ell=0}(z,\tau) + 2 \sum_{n\geq 0} A(n) ch^{\tilde{R}}_{h=\frac{1}{4}+n,\ell=\frac{1}{2}}(z,\tau)$$

At smaller values of $n$, Fourier coefficients $A(n)$ may be obtained by direct expansion. We find, $A(0) = -1$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 $\cdots$ |
|---|---|---|---|---|---|---|---|---|
| $A(n)$ | 45 | 231 | 770 | 2277 | 5796 | 13915 | 30843 | 65550 $\ldots$ |

Surprize: Dimensions of some irreducible reps. of Mathieu group $M_{24}$ appear

dimensions : $\{$ 45 231 770 990 1771 2024 2277 3312 3520 5313 5544 5796 10395 $\cdots\}$

$$A(6) = 13915 = 3520 + 10395,$$

$$A(7) = 30843 = 10395 + 5796 + 5544 + 5313 + 2024 + 1771$$

Mathieu moonshine? T.E.-Ooguri-Tachikawa

cf. Monsterous moonshine:

$$J(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots$$

q-expansion coeffcients of J-function are decomposed into a sum of irred. reps. of the monster group.

$$196884 = 1 + 196883, \ \ 21493760 = 1 + 196883 + 21296876$$

Mukai: enumeration of eleven K3 surfaces with finite non-Abelian automorphism group. All these groups are sugbgroups of $M_{23}$.

Fantasy: Is it possible that these automorphism groups at isoletd points in K3 moduli space are enhanced to $M_{24}$ over the whole moduli space when one cosnider the elliptic genus?

On the other hand, using the method of Rademacher expansion adapted to the case of Mock theta functions ( Bringmann-Ono ) we can determine the asymptotic behavior of coefficients $A(n)$ as

$$A(n) \approx \frac{2}{\sqrt{8n-1}} e^{2\pi\sqrt{\frac{1}{2}(n-\frac{1}{8})}}$$

T.E.-Hikami

♣ Twisted Elliptic Genus

Dimension of the representation equals the trace of the identity element: we may identify

$$A(n) = Tr_{V_n}\ 1$$
$$V_1 = 45 + 45^*,\ V_2 = 231 + 231^*,\ V_3 = 770 + 770^*, \cdots$$

We consider the trace of other group elements in $M_{24}$

$$A_g(n) = Tr_{V_n}\ g, \qquad g \in M_{24}$$

$Tr\ g$ depends only on the conjugacy class of $g$. There exists 26 conjugacy classes $\{g\}$ in $M_{24}$ and also 26 irreducible representations $\{R\}$. We have the character table given by

$$\chi_R{}^g = Tr_R\ g$$

| 1A | 2A | 3A | 5A | 4B | 7A | 7B | 8A | 6A | 11A | 15A | 15B | 14A | 14B | 23A | 23B | 12B | 6B | 4C | 3B |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 23 | 7 | 5 | 3 | 3 | 2 | 2 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | -1 | -1 | -1 |
| 252 | 28 | 9 | 2 | 4 | 0 | 0 | 0 | 1 | -1 | -1 | -1 | 0 | 0 | -1 | -1 | 0 | 0 | 0 | 0 |
| 253 | 13 | 10 | 3 | 1 | 1 | 1 | -1 | -2 | 0 | 0 | 0 | -1 | -1 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1771 | -21 | 16 | 1 | -5 | 0 | 0 | -1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | -1 | -1 | -1 | 7 |
| 3520 | 64 | 10 | 0 | 0 | -1 | -1 | 0 | -2 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | -8 |
| 45 | -3 | 0 | 0 | 1 | $e_7^+$ | $e_7^-$ | -1 | 0 | 1 | 0 | 0 | $-e_7^+$ | $-e_7^-$ | -1 | -1 | 1 | -1 | 1 | 3 |
| $\overline{45}$ | -3 | 0 | 0 | 1 | $e_7^-$ | $e_7^+$ | -1 | 0 | 1 | 0 | 0 | $-e_7^-$ | $-e_7^+$ | -1 | -1 | 1 | -1 | 1 | 3 |
| 990 | -18 | 0 | 0 | 2 | $e_7^+$ | $e_7^-$ | 0 | 0 | 0 | 0 | 0 | $e_7^+$ | $e_7^-$ | 1 | 1 | 1 | -1 | -2 | 3 |
| $\overline{990}$ | -18 | 0 | 0 | 2 | $e_7^-$ | $e_7^+$ | 0 | 0 | 0 | 0 | 0 | $e_7^-$ | $e_7^+$ | 1 | 1 | 1 | -1 | -2 | 3 |
| 1035 | -21 | 0 | 0 | 3 | $2\,e_7^+$ | $2\,e_7^-$ | -1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 1 | -1 | -3 |
| $\overline{1035}$ | -21 | 0 | 0 | 3 | $2\,e_7^-$ | $2\,e_7^+$ | -1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 1 | -1 | -3 |
| $1035'$ | 27 | 0 | 0 | -1 | -1 | -1 | 1 | 0 | 1 | 0 | 0 | -1 | -1 | 0 | 0 | 0 | 2 | 3 | 6 |
| 231 | 7 | -3 | 1 | -1 | 0 | 0 | -1 | 1 | 0 | $e_{15}^+$ | $e_{15}^-$ | 0 | 0 | 1 | 1 | 0 | 0 | 3 | 0 |
| $\overline{231}$ | 7 | -3 | 1 | -1 | 0 | 0 | -1 | 1 | 0 | $e_{15}^-$ | $e_{15}^+$ | 0 | 0 | 1 | 1 | 0 | 0 | 3 | 0 |
| 770 | -14 | 5 | 0 | -2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | $e_{23}^+$ | $e_{23}^-$ | 1 | 1 | -2 | -7 |
| $\overline{770}$ | -14 | 5 | 0 | -2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | $e_{23}^-$ | $e_{23}^+$ | 1 | 1 | -2 | -7 |
| 483 | 35 | 6 | -2 | 3 | 0 | 0 | -1 | 2 | -1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 0 |
| 1265 | 49 | 5 | 0 | 1 | -2 | -2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -3 | 8 |
| 2024 | 8 | -1 | -1 | 0 | 1 | 1 | 0 | -1 | 0 | -1 | -1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 8 |
| 2277 | 21 | 0 | -3 | 1 | 2 | 2 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | -3 | 6 |
| 3312 | 48 | 0 | -3 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | -1 | -1 | 0 | 0 | 0 | -2 | 0 | -6 |
| 5313 | 49 | -15 | 3 | -3 | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -3 | 0 |
| 5796 | -28 | -9 | 1 | 4 | 0 | 0 | 0 | -1 | -1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5544 | -56 | 9 | -1 | 0 | 0 | 0 | 0 | 1 | 0 | -1 | -1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 10395 | -21 | 0 | 0 | -1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | -1 | 0 | 0 | 3 | 0 |

Here we have used $e_p^{\pm} = \frac{1}{2}\left(\pm\sqrt{-p} - 1\right)$.

There are two types of conjugacy classes in $M_{24}$, type I and type II.

Conjugacy class of type I fixes at least one element out of 24 and thus they arise from the conjugacy classes of $M_{23}$.

On the other hand conjugacy class of type II does not have a fixed point and is intrinsically $M_{24}$.

For each conjugacy class we want to construct a twisted genus (analogue of Thompson series in monstrous moonshine)

$$A_g = \sum_{n=1}^{\infty} Tr_{V_n}\, g \times q^n$$

For instance,

$$A_{2A} = -6q + 14q^2 - 28q^3 + 42q^4 - 56q^5 + 86q^6 + \cdots$$

and has the right modular property ($Z_{2A} \in \Gamma_0(2)$).

Twisted genus is decomposed into massless and massive parts

$$Z_g(\tau, z) = \chi_g\, ch^{\tilde{R}}_{h=\frac{1}{4},\ell=0} + \sum_{n\geq 0} A_g(n) ch^{\tilde{R}}_{\frac{1}{4}+n,\ell=\frac{1}{2}}(z,\tau)$$

Here $\chi_g$ is the Euler number assigned to the class $g$

| $g$ | 1A | 2A | 3A | 5A | 4B | 7A | 8A | 6A | 11A | 15A | 14A | 23A | others |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\chi_g$ | 24 | 8 | 6 | 4 | 4 | 3 | 2 | 2 | 2 | 1 | 1 | 1 | 0 |

$\chi_g$ vanishes for type II classes. We note that $\chi_g$ can be written as $\chi_g = \chi_1^g + \chi_{23}^g$ which is equal to the number of fixed points of the permutation rep. of $g$.

| conjugacy class | cycle shape |
|---|---|
| 1A | $1^{24}$ |
| 2A | $1^8 \cdot 2^8$ |
| 3A | $1^6 \cdot 3^6$ |
| 5A | $1^4 \cdot 5^4$ |
| 4B | $1^4 \cdot 2^2 \cdot 4^4$ |
| 7A | $1^3 \cdot 7^3$ |
| 7B | $1^3 \cdot 7^3$ |
| 8A | $1^2 \cdot 2^1 \cdot 4^1 \cdot 8^2$ |
| 6A | $1^2 \cdot 2^2 \cdot 3^2 \cdot 6^2$ |
| 11A | $1^2 \cdot 11^2$ |
| 15A | $1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$ |
| 15B | $1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$ |
| 14A | $1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$ |
| 14B | $1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$ |
| 23A | $1^1 \cdot 23^1$ |
| 23B | $1^1 \cdot 23^1$ |
| 12B | $12^2$ |
| 6B | $6^4$ |
| 4C | $4^6$ |
| 3B | $3^8$ |
| 2B | $2^{12}$ |
| 10A | $2^2 \cdot 10^2$ |
| 21A | $3^1 \cdot 21^1$ |
| 21B | $3^1 \cdot 21^1$ |
| 4A | $2^4 \cdot 4^4$ |
| 12A | $2^1 \cdot 4^1 \cdot 6^1 \cdot 12^1$ |

Twisted genera for all conjugacy classes have been obtained by collective efforts by various authors. They reproduce correct lower-order expansion coefficients and are invariant under the Hecke subgroup $\Gamma_0(N)$

$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; ad - bc = 1, \; c \equiv 0, \; mod\, N\}$$

$N$ denotes the order of the element $g$.

M.Cheng, Gaberdiel,Hohenegger and Volpato, T.E. and K.Hikami

From the study of K3 surface with $Z_p (p = 2, 3, \cdots)$ symmetry, for instance, twisted genera of classes

$pA(p = 2, 3, \cdots)$ are known A.Sen

$$Z_{pA}(z;\tau) = \frac{2}{p+1}\phi_{0,1}(z;\tau) + \frac{2p}{p+1}\phi_2^{(p)}(\tau)\phi_{-2,1}(z;\tau)$$

where

$$\phi_{0,1}(z;\tau) = \frac{1}{2}Z_{K3}(z;\tau), \quad \phi_{-2,1}(z;\tau) = -\frac{\theta_1(z;\tau)^2}{\eta(\tau)^6}$$

are the basis of Jacobi forms with index=1 and

$$\phi_2^{(p)}(\tau) = \frac{24}{p-1} q\partial_q \log\left(\frac{\eta(p\tau)}{\eta(\tau)}\right),$$
$$= \frac{24}{p-1}\sum_{k=1}\sigma_1(k)(q^k - pq^{pk})$$

is an element of $\Gamma_0(p)$.

In the case of type II twisted genera are modular forms of $\Gamma_0(N)$ with a multiplier system (invariant up to a phase). They are given in terms of quotients of

eta functions.

$$Z_{2B}(z;\tau) = 2\frac{\eta(\tau)^8}{\eta(2\tau)^4}\phi_{-2,1}(z;\tau),$$
$$Z_{3B}(z;\tau) = 2\frac{\eta(\tau)^6}{\eta(3\tau)^2}\phi_{-2,1}(z;\tau),$$
$$Z_{4A}(z;\tau) = 2\frac{\eta(2\tau)^8}{\eta(4\tau)^4}\phi_{-2,1}(z;\tau),$$
$$Z_{4C}(z;\tau) = 2\frac{\eta(\tau)^4\eta(2\tau)^2}{\eta(4\tau)^2}\phi_{-2,1}(z;\tau)$$
$$\cdots$$

etc. The multiplier sytem has been studied in detail by Gaberdiel, Persson, Ronellenfitsch and Volpato . Thus we have a complete list of the twisted genera for 26 conjugacy classes. Making use of them we can uniquely decompose the coefficients of K3 elliptic genus into irreducible representations of $M_{24}$ at arbitrary level.

| n | 1A | 2A | 3A | 5A | 4B | 7A | 8A | 6A | 11A | 15A | 14A | 23A | 12B | 6B | 4C | 3B |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 90 | -6 | 0 | 0 | 2 | -1 | -2 | 0 | 2 | 0 | 1 | -2 | 2 | -2 | 2 | 6 |
| 2 | 462 | 14 | -6 | 2 | -2 | 0 | -2 | 2 | 0 | -1 | 0 | 2 | 0 | 0 | 6 | 0 |
| 3 | 1540 | -28 | 10 | 0 | -4 | 0 | 0 | 2 | 0 | 0 | 0 | -1 | 2 | 2 | -4 | -14 |
| 4 | 4554 | 42 | 0 | -6 | 2 | 4 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | -6 | 12 |
| 5 | 11592 | -56 | -18 | 2 | 8 | 0 | 0 | -2 | -2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 27830 | 86 | 20 | 0 | -2 | -2 | 2 | -4 | 0 | 0 | 2 | 0 | 0 | 0 | 6 | -16 |
| 7 | 61686 | -138 | 0 | 6 | -10 | 2 | -2 | 0 | -2 | 0 | 2 | 0 | -2 | -2 | -2 | 30 |
| 8 | 131100 | 188 | -30 | 0 | 4 | -3 | 0 | 2 | 2 | 0 | -1 | 0 | 0 | 0 | -12 | 0 |
| 9 | 265650 | -238 | 42 | -10 | 10 | 0 | -2 | 2 | 0 | 2 | 0 | 0 | -2 | 6 | 10 | -42 |
| 10 | 521136 | 336 | 0 | 6 | -8 | 0 | -4 | 0 | 0 | 0 | 0 | 2 | -2 | 2 | 16 | 42 |
| 11 | 988770 | -478 | -60 | 0 | -14 | 6 | 2 | -4 | 2 | 0 | -2 | 0 | 0 | 0 | -6 | 0 |
| 12 | 1830248 | 616 | 62 | 8 | 8 | 0 | 0 | -2 | 2 | 2 | 0 | 0 | 2 | -6 | -16 | -70 |
| 13 | 3303630 | -786 | 0 | 0 | 22 | -6 | 2 | 0 | 0 | 0 | -2 | 2 | 0 | -4 | 6 | 84 |
| 14 | 5844762 | 1050 | -90 | -18 | -6 | 0 | 2 | 6 | 0 | 0 | 0 | 2 | 0 | 0 | 18 | 0 |
| 15 | 10139734 | -1386 | 118 | 4 | -26 | -4 | -2 | 6 | 0 | -2 | 0 | 0 | 2 | 2 | -10 | -110 |
| 16 | 17301060 | 1764 | 0 | 0 | 12 | 0 | 0 | 0 | -4 | 0 | 0 | 0 | 2 | 6 | -28 | 126 |
| 17 | 29051484 | -2212 | -156 | 14 | 28 | 0 | -4 | -4 | 0 | -1 | 0 | 0 | 0 | 0 | 12 | 0 |
| 18 | 48106430 | 2814 | 170 | 0 | -18 | 8 | -2 | -6 | -2 | 0 | 0 | -2 | 2 | -6 | 38 | -166 |
| 19 | 78599556 | -3612 | 0 | -24 | -36 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | -2 | -6 | -20 | 210 |
| 20 | 126894174 | 4510 | -228 | 14 | 14 | -6 | -2 | 4 | 0 | 2 | 2 | 0 | 0 | 0 | -42 | 0 |
| 21 | 202537080 | -5544 | 270 | 0 | 48 | 4 | 4 | 6 | -2 | 0 | 0 | 0 | -2 | 6 | 16 | -282 |
| 22 | 319927608 | 6936 | 0 | 18 | -16 | -7 | 4 | 0 | 0 | 0 | -1 | 0 | 0 | 4 | 48 | 300 |
| 23 | 500376870 | -8666 | -360 | 0 | -58 | 0 | -2 | -8 | 4 | 0 | 0 | 2 | 0 | 0 | -18 | 0 |
| 24 | 775492564 | 10612 | 400 | -36 | 28 | 0 | 0 | -8 | 0 | 0 | 0 | 0 | 0 | -8 | -60 | -392 |
| 25 | 1191453912 | -12936 | 0 | 12 | 64 | 12 | -4 | 0 | 0 | 0 | 0 | 0 | 2 | -10 | 32 | 462 |
| 26 | 1815754710 | 15862 | -510 | 0 | -34 | 0 | -6 | 10 | 0 | 0 | 0 | -1 | 0 | 0 | 78 | 0 |
| 27 | 2745870180 | -19420 | 600 | 30 | -76 | -10 | 4 | 8 | -2 | 0 | -2 | 0 | 0 | 8 | -36 | -600 |
| 28 | 4122417420 | 23532 | 0 | 0 | 36 | 2 | 0 | 0 | 0 | 0 | -2 | 0 | 0 | 12 | -84 | 660 |
| 29 | 6146311620 | -28348 | -762 | -50 | 100 | -6 | 4 | -10 | -2 | -2 | 2 | 0 | 0 | 0 | 36 | 0 |
| 30 | 9104078592 | 34272 | 828 | 22 | -40 | 0 | 4 | -12 | 4 | -2 | 0 | 0 | 0 | -8 | 96 | -840 |
| 31 | 13401053820 | -41412 | 0 | 0 | -116 | 0 | -4 | 0 | 0 | 0 | 0 | -2 | -2 | -10 | -44 | 966 |
| 32 | 19609321554 | 49618 | -1062 | 34 | 50 | 18 | 2 | 10 | -2 | -2 | 2 | 0 | 0 | 0 | -126 | 0 |
| 33 | 28530824630 | -59178 | 1220 | 0 | 126 | 0 | -6 | 12 | 0 | 0 | 0 | 2 | -4 | 12 | 62 | -1204 |
| 34 | 41286761478 | 70758 | 0 | -72 | -66 | -10 | -6 | 0 | 6 | 0 | 2 | 0 | 0 | 12 | 150 | 1332 |
| 35 | 59435554926 | -84530 | -1518 | 26 | -154 | 6 | 2 | -14 | 0 | 2 | 2 | 0 | 0 | 0 | -66 | 0 |
| 36 | 85137361430 | 100310 | 1670 | 0 | 70 | -12 | -2 | -10 | 0 | 0 | 0 | 0 | -2 | -18 | -170 | -1666 |

♣ Proof of Mathieu moonshine

Orthogonality relation of characters:

$$\sum_g n_g \chi_{R'}^g \bar{\chi}_R^g = |G|\delta_{RR'}$$

$n_g$ is the number of elements in the conjugacy class $g$ and $|G|$ denotes the order of the group. Let $c_R(n)$ be he multiplicity of representation $R$ in the decom-postion of K3 elliptic genus at level $n$. We then have

$$\sum_R c_R(n)\chi_R^g = A_g(n)$$

Then using the orthogonality relation we find

$$\sum_g \frac{1}{|G|} n_g \bar{\chi}_R^g A_g(n) = c_R(n)$$

We have checked that the multiplicities $c_R(n)$ are all positive integers upto $n = 1000$ and this gives a very strong evidence for Mathieu moonshine con-jecture.

| n | 1 | 23 | 252 | 253 | 1771 | 3520 | $\frac{45}{45}$ | $\frac{990}{990}$ | $\frac{1035}{1035}$ | $1035'$ | $\frac{231}{231}$ | $\frac{770}{770}$ | 483 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 2 | 4 | 0 | 0 | 2 | 2 | 0 | 2 | 2 |
| 10 | 0 | 0 | 0 | 2 | 4 | 8 | 0 | 2 | 2 | 2 | 2 | 0 | 2 |
| 11 | 0 | 0 | 0 | 0 | 8 | 12 | 0 | 4 | 4 | 6 | 0 | 4 | 0 |
| 12 | 0 | 2 | 2 | 4 | 12 | 30 | 0 | 8 | 8 | 4 | 2 | 6 | 4 |
| 13 | 0 | 0 | 4 | 2 | 26 | 44 | 2 | 14 | 14 | 18 | 2 | 10 | 6 |
| 14 | 0 | 0 | 4 | 6 | 38 | 86 | 0 | 24 | 24 | 22 | 8 | 16 | 14 |
| 15 | 0 | 0 | 12 | 8 | 78 | 144 | 2 | 40 | 44 | 46 | 8 | 38 | 18 |
| 16 | 0 | 2 | 18 | 22 | 122 | 252 | 2 | 72 | 72 | 68 | 18 | 50 | 36 |
| 17 | 0 | 2 | 30 | 26 | 212 | 410 | 8 | 116 | 124 | 130 | 25 | 94 | 54 |
| 18 | 0 | 6 | 50 | 58 | 342 | 704 | 6 | 194 | 202 | 192 | 50 | 148 | 100 |
| 19 | 0 | 4 | 80 | 72 | 582 | 1116 | 18 | 318 | 332 | 346 | 68 | 252 | 150 |
| 20 | 0 | 14 | 128 | 138 | 904 | 1836 | 20 | 516 | 536 | 520 | 126 | 390 | 254 |
| 21 | 2 | 20 | 214 | 200 | 1476 | 2902 | 40 | 814 | 860 | 872 | 182 | 652 | 396 |
| 22 | 2 | 32 | 328 | 346 | 2302 | 4616 | 55 | 1298 | 1348 | 1336 | 314 | 988 | 640 |
| 23 | 2 | 40 | 512 | 496 | 3638 | 7166 | 98 | 2020 | 2118 | 2144 | 460 | 1590 | 972 |
| 24 | 0 | 80 | 798 | 824 | 5584 | 11192 | 132 | 3140 | 3278 | 3236 | 744 | 2426 | 1544 |
| 25 | 8 | 108 | 1232 | 1208 | 8654 | 17084 | 234 | 4814 | 5038 | 5084 | 1106 | 3764 | 2336 |
| 26 | 6 | 174 | 1860 | 1904 | 13090 | 26148 | 322 | 7348 | 7670 | 7626 | 1742 | 5677 | 3602 |
| 27 | 12 | 252 | 2836 | 2802 | 19914 | 39436 | 514 | 11092 | 11618 | 11666 | 2560 | 8688 | 5394 |
| 28 | 16 | 398 | 4238 | 4310 | 29772 | 59330 | 742 | 16686 | 17418 | 17356 | 3922 | 12912 | 8160 |
| 29 | 26 | 560 | 6328 | 6286 | 44512 | 88280 | 1154 | 24840 | 25994 | 26078 | 5758 | 19380 | 12090 |
| 30 | 34 | 876 | 9368 | 9486 | 65776 | 131020 | 1642 | 36824 | 38480 | 38368 | 8642 | 28580 | 18008 |
| 31 | 58 | 1236 | 13802 | 13764 | 97060 | 192538 | 2500 | 54178 | 56660 | 56800 | 12582 | 42218 | 26384 |
| 32 | 76 | 1866 | 20166 | 20356 | 141714 | 282074 | 3564 | 79320 | 82884 | 82730 | 18576 | 61574 | 38738 |
| 33 | 122 | 2664 | 29396 | 29374 | 206524 | 410062 | 5286 | 115334 | 120644 | 120798 | 26830 | 89868 | 56226 |
| 34 | 166 | 3900 | 42474 | 42810 | 298508 | 593800 | 7542 | 166990 | 174510 | 174330 | 39066 | 129694 | 81546 |
| 35 | 248 | 5536 | 61184 | 61234 | 430134 | 854284 | 10988 | 240304 | 251292 | 251544 | 55956 | 187094 | 117138 |

**Recently Gannon has proved by mathematical induction that the multiplicities are all positive integers for all $n$.**

♣ Recent Developments

**Umbral moonshine:** Cheng, Duncan and Harvey

Consider a series of Jacobi forms with index $k$ ($k =$

$1, 2, 3, 4, 6$)

$$Z(k=1) = 8\,(f_2 + f_3 + f_4),$$
$$Z(k=2) = 4\,(f_2 f_3 + f_3 f_4 + f_4 f_2),$$
$$Z(k=3) = 8\, f_2 f_3 f_4,$$
$$Z(k=4) = 4\,(f_2^2 f_3 f_4 + \cdots) - 2\,(f_2^2 f_3^2 + \cdots),$$
$$Z(k=6) = -4\,(f_2^3 f_3^3 + \cdots) + 4\,(f_2^3 f_3^2 f_4 + \cdots) - 8\, f_2^2 f_3^2 f_4^2$$

where $f_2 = \left(\dfrac{\theta_2(z)}{\theta_2(0)}\right)^2, f_3 = \left(\dfrac{\theta_3(z)}{\theta_3(0)}\right)^2, f_4 = \left(\dfrac{\theta_4(z)}{\theta_4(0)}\right)^2$

These Jacobi forms are characterized by their $q^0$

term

$$Z(k) \approx 2y + (\frac{24}{k} - 4) + 2y^{-1}, \quad y = e^{2\pi i z}$$

It turns out that the expansion of the above Jacobi forms in terms of $\mathcal{N} = 4$ characters all exhibit moonshine phenomena, with the group $M_{24}$ for $k = 1$, $2.M_{12}$ for $k = 2$, $2.AGL_3(2)$ for $k = 3$ etc.

Note:
At $k = 2$, for instance, there exist two Jacobi forms with index 2

$$J_1 = f_2^2 + f_3^2 + f_4^2, \quad J_2 = f_2 f_3 + f_3 f_4 + f_4 f_2$$

It is known that the identity operator in NS sector is contained in $J_1$. The elliptic genus of symmetric product $K3^{[2]}$, for instance, is given by

$$48J_1 + 60J_2.$$

It is somewhat awkward to consider the choice $Z(k = 2) = 4J_2$ which does not contain the identity operator. Thus $Z(k = 2)$ may not possess well-defined geometrical significance. The same comment applies to all cases $k \geq 2$.

Umbral moonshine series, however, appears to give a natural extension of original Mathieu moonshine.

Very recently its relation to Niemeier lattice which are self-dual lattices in 24 dimensions has been discovered. If the lattice has an automorphism group $G$, then it turns out that the group $G/W$ becomes the symmetry group of the Moch modular form where $W$ denotes the Weyl group of the root lattice. Above 5 cases correspond to Niemeier lattices based on powers of SU(N) type root lattices, $24A_1, 12A_2, 8A_3$ etc.

N=2 moonshine T.E. and Hikami

$$
\begin{aligned}
Z(k=2) &= \text{massless}(N=2, Q=0) \\
&+ \sum_n H_1(n)\,\text{massive}(N=2, Q=\pm 1) \\
&+ \sum_n H_2(n)\,\text{massive}(N=2, Q=\pm 2)
\end{aligned}
$$

$H_1, H_2$ are decomposed into sums of representations of group $SL_2(11)$.

Summary

- There is a strong evidence for Mathieu moonshine phenomenon for K3 surface.

• It is beyond classical geometry and no fundamental explanations so far.

• Hilbert space of string theory compactified on K3 surfaces does not possess symmetry under $M_{24}$. Gaberdiel-Hohenegger-Volpato, Taormina-Wendland. Rather the $M_{24}$ symmetry should be searched in the BPS or topololgical sector of the theory.

• Umbral moonshine gives a natural generalization of Mathieu moonshine although its geometrical significance is yet somewhat obscure.

♠ Golay code

Mathieu group $M_{24}$ is a symmetry of the Golay code: we introduce binary numbers of 24 digits and select 759 octads (8 up spins) and 2576 dodecads (12 up spins) in such a way that any pair of these have a distance larger than 8. We have the decomposition of the subspace of dimension $2^{12}$

$$1 + 759 + 2576 + 759 + 1 = 4096 = 2^{12}$$

This is the Golay code. Mathieu group $M_{24}$ is the symmetry of the Goley code and is quintuply transitive. (An arbitrary chosen 5 elements belong to a unique octad)

$$24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 = 759 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$$

♠ Zweger's method

The above function $\mu$ has the S-tranformation

$$\mu(z;\tau) + \sqrt{\frac{i}{\tau}}\mu(\frac{z}{\tau}; -\frac{1}{\tau}) = \frac{1}{2}M(\tau)$$

$$M(\tau) = \int_{-\infty}^{+\infty} \frac{e^{\pi i \tau z^2}}{\cosh \pi z} dz$$

⇑

Modell integral

Zwegers prescription to cure the modular property is to first introduce a non-holomrophic partner of $\mu(z;\tau)$

$$R(\tau) = \sum(-1)^n[sgn(n+\frac{1}{2}) - E((n+\frac{1}{2})\sqrt{2\tau_2})] \times q^{-\frac{1}{2}(n+\frac{1}{2})^2}, \quad \tau = \tau_1 + i\tau_2$$

Here $E$ denotes the error function. By construction $R(\tau)$ obeys a transformation law

$$R(\tau) + \sqrt{\frac{i}{\tau}}R(-\frac{1}{\tau}) = M(\tau)$$

Then we form a combination

$$\hat{\mu}(z;\tau) \equiv \mu(z;\tau) - \frac{1}{2}R(\tau)$$

The Mordell integral cancels out and $\hat{\mu}$ has a good modular property

$$\hat{\mu}(z;\tau) = -\sqrt{\frac{i}{\tau}}\hat{\mu}(\frac{z}{\tau};-\frac{1}{\tau})$$

and is in fact an analytic Jacobi form of weight=1/2 and index=0. Explicitly $R$ is expressed by a contour integral

$$R(\tau) = -i\int_{-\bar{\tau}}^{i\infty} \frac{\eta(z)^3}{\sqrt{\frac{z+\tau}{i}}}dz$$

(Character of non-BPS representations of general level k N=4 algebra is given by;

$$q^{h-\frac{(\ell+\frac{1}{2})^2}{k+1}-\frac{k}{4}}\chi_{k-1,\ell}\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}$$

At unitarity boundary ($h=k/4$) it splits into a sum of massless reps.

$$q^{-\frac{(\ell+\frac{1}{2})^2}{k+1}}\chi_{k-1,\ell}\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3} = ch^{\tilde{R}}_{k,\ell+\frac{1}{2}} + 2ch^{\tilde{R}}_{k,\ell} + ch^{\tilde{R}}_{k,\ell-\frac{1}{2}} \quad )$$

Now consider

$$\begin{aligned} J(z,w;\tau) &= \frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}\left(\hat{\mu}(z;\tau)-\hat{\mu}(w;\tau)\right) \\ &= \frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}\left(\mu(z;\tau)-\mu(w;\tau)\right) \\ &= ch^{\tilde{R}}_{k=1,h=\frac{1}{4},\ell=0}(z;\tau)-\mu(w;\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3} \end{aligned}$$

By construction

$$J(w,w;\tau)=0, \qquad J(z=0,w;\tau)=1$$

and $J$ is a holomorphic Jacobi form of weight=0

and index=1. We can determine them as

$$\begin{aligned} J(z,w=\frac{1}{2};\tau) &= \left(\frac{\theta_2(z;\tau)}{\theta_2(0;\tau)}\right)^2, \\ J(z,w=\frac{1+\tau}{2};\tau) &= \left(\frac{\theta_3(z;\tau)}{\theta_3(0;\tau)}\right)^2, \\ J(z,w=\frac{\tau}{2};\tau) &= \left(\frac{\theta_4(z;\tau)}{\theta_4(0;\tau)}\right)^2 \end{aligned}$$

## $\mathcal{N} = 4$ superconformal algebra

$$
\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \frac{k}{2}(m^3 - m)\delta_{m+n,0} \\
&\{G^a_r, G^b_s\} = \{G^{*a}_{\ r}, G^{*b}_{\ s}\} = 0 \qquad a, b = 1, 2 \\
&\{G^a_r, \bar{G}^b_s\} = 2\delta^{ab}L_{r+s} - 2(r-s)\sigma^i_{ab}J^i_{r+s} + \frac{k}{2}(4r^2 - 1)\delta_{r+s}\delta^{ab} \\
&[J^i_m, J^j_n] = i\epsilon^{ijk}J^k_{m+n} + \frac{k}{2}m\delta_{m+n,0}\delta^{ij} \\
&[J^i_m, G^a_r] = -\frac{1}{2}\sigma^i_{ab}G^b_{m+r}, \quad [J^i_m, G^{*a}_{\ r}] = \frac{1}{2}\sigma^{i*}_{ab}G^{*b}_{\ m+r}, \\
&[L_m, G^a_r] = (\frac{m}{2} - r)G^a_{m+r}, \quad [L_m, G^{*a}_{\ r}] = (\frac{m}{2} - r)G^{*a}_{\ m+r} \\
&[L_m, J^i_n] = -nJ^i_{m+n}
\end{aligned}
$$

$$c = 6k$$

### Automorphism

$$
\begin{aligned}
&L'_n = L_n + 2\eta J^3_n + \eta^2\,\frac{c}{6}\,\delta_{n,0} \\
&J'^{\pm}_n = J^{\pm}_{n+2\eta} \\
&J'^3_n = J^3_n + \eta\,\frac{c}{6}\,\delta_{n,0} \\
&G^{1,2'}_r = G^{1,2'}_{r\mp\eta} \\
&G^{*1,2'}_{\ r} = G^{*1,2'}_{\ r\pm\eta}
\end{aligned}
$$

## Completion

$$\widehat{f_u^{(k)}}(\tau,z) = f_u^{(k)}(\tau,z) - \frac{1}{2}\sum_{m\in \mathrm{Z}_{2k}} R_{m,k}(\tau,u)\Theta_{m,k}(\tau,2z), (k\in \mathrm{Z}>0)$$

$$f_u^{(k)}(\tau,z) = \sum_{n\in Z}\frac{q^{kn^2}y^{2kn}}{1-yw^{-1}q^n}, (q=e^{2\pi i\tau},\ y=e^{2\pi iz},\ w=e^{2\pi iu})$$

$$R_{m,k}(\tau,u)$$
$$= \sum_{\nu\in m+2k\mathrm{Z}}\left[\mathrm{sgn}(\nu+0)-\mathrm{Erf}\left\{\sqrt{\frac{\pi\tau_2}{k}}\left(\nu+2k\frac{u_2}{\tau_2}\right)\right\}\right] w^{-\nu}q^{-\frac{\nu^2}{4k}}$$
$$(\tau_2=\mathrm{Im}\tau, u_2=\mathrm{Im}u)$$

**Error function**

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

**Integral representation**

$$\mathrm{sgn}(\nu+0)-\mathrm{Erf}(\nu) = \frac{1}{i\pi}\int_{\mathrm{R}-i0} dp\frac{e^{-(p^2+\nu^2)}}{p-i\nu}$$

**Poincare series representation (we set $u=0$)**

$$\widehat{f}^{(k)}(\tau,z) = \frac{i}{2\pi}\sum_{n,m\in\mathrm{Z}} q^{km^2}y^{2km}\frac{e^{-\frac{\pi k}{\tau_2}(z+m\tau+n)^2}}{z+m\tau+n}$$