極小表現の解析

Geometric Analysis on Minimal Representations

第 9 回 岡シンポジュウム, 奈良女子大学, 2010 年 12 月 4 日

小林俊行
(東京大学)

http://www.ms.u-tokyo.ac.jp/~toshi/

What are minimal reps?

Minimal representations of a reductive group G

Algebraically, minimal reps are infinite dim'l reps whose annihilators are the Joseph ideals in $U(\mathfrak{g})$

Loosely, minimal representations are

- 'smallest' infinite dimensional unitary rep. of G
- one of 'building blocks' of unitary reps.
- 'isolated' among the unitary dual
  (finitely many) (continuously many)
- 'attached to' minimal nilpotent orbits (orbit method)
- matrix coefficients are of bad decay

Building blocks of unitary reps

unitary reps of Lie groups
↑ direct integral (Mautner)
irred. unitary reps of Lie groups
↑ construction (Mackey, Kirillov, Duflo)
irred. unitary reps of reductive groups
↑ “induction”, etc.
finitely many “very small” irred. unitary reps.
of reductive groups
(e.g. 1 dim’l trivial rep., minimal rep, etc.)

Building blocks of unitary reps

unitary reps of Lie groups
↑ direct integral (Mautner)
irred. unitary reps of Lie groups
↑ construction (Mackey, Kirillov, Duflo)

Cf. Orbit philosophy

Jordan normal forms
↑ semisimple matrices
finitely many types of nilpotent matrices

Orbit philosophy

Orbit philosophy à la Kirillov–Kostant–Duflo

$G \overset{\mathrm{Ad}^*}{\curvearrowright} \mathfrak{g}^*$ coadjoint action

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Orbit philosophy

Orbit philosophy à la Kirillov–Kostant–Duflo

$$\boxed{\mathfrak{g}^*/\,\mathrm{Ad}^*(G) \quad \doteqdot \quad \widehat{G} \quad \text{(unitary dual)}}$$

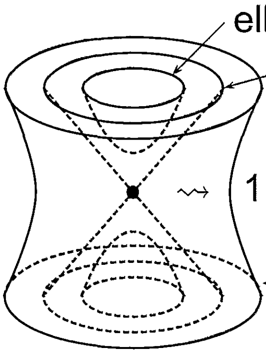
works perfectly for nilpotent group G
not work perfectly for reductive group G
(still open)

Orbit philosophy

Orbit philosophy à la Kirillov–Kostant–Duflo

$$\boxed{\mathfrak{g}^*/\operatorname{Ad}^*(G) \quad \doteqdot \quad \widehat{G} \quad \text{(unitary dual)}}$$

“geometric quantization”



elliptic orbit $\rightsquigarrow$ discrete series

nilpotent orbit $\rightsquigarrow$ $\boxed{\ ?\ }$

$\rightsquigarrow$ 1 dim’l trivial rep

hyperbolic orbit $\rightsquigarrow$ principal series

$G = SL(2,\mathbb{R})$

minimal nilpotent orbits $\rightsquigarrow$ minimal reps?

Minimal representations

Oscillator rep. (= Segal–Shale–Weil rep.)
Minimal rep. of $Mp(n,\mathbb{R})$ (= double cover of $Sp(n,\mathbb{R})$)
$\cdots$ split simple group of type C

Today: Geometric and analytic aspects of
Minimal rep. of $O(p,q)$, $p+q$: even
$\cdots$ simple group of type D

Cf. There is no minimal rep of simple group of type A
Minimal rep. of $O(p,q)$, $p+q$: odd, $p,q>3$ does not exist.
$\cdots$ simple group of type B

One does not know “canonical” construction of minimal representations

Minimal representations

Oscillator rep. (= Segal–Shale–Weil rep.)
Minimal rep. of $Mp(n, \mathbb{R})$ (= double cover of $Sp(n, \mathbb{R})$)
$\cdots$ split simple group of type C

Today: Geometric and analytic aspects of
Minimal rep. of $O(p, q)$, $p + q$: even
$\cdots$ simple group of type D

(Ambitious) Project: ([K– , to appear])
Use minimal reps to get an inspiration in finding new interactions with other fields of mathematics.

If possible, try to formulate a theory in a wide setting without group, and prove it without representation theory.

What are minimal reps?

Minimal representations of a reductive group G

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- matrix coefficients are of bad decay

Minimal $\Leftrightarrow$ Maximal

(Ambitious) Project: ([1])
Use minimal reps to get an inspiration in finding new interactions with other fields of mathematics.

Observation. ϖ: minimal rep of G
$\mathrm{DIM}(\varpi)$ (Gelfand–Kirillov dimension)
$= \frac{1}{2}$ dimension of minimal nilpotent orbits
$<$ dimension of any non-trivial G-space

Minimal $\Leftrightarrow$ Maximal

(Ambitious) Project: ([1])
Use minimal reps to get an inspiration in finding new interactions with other fields of mathematics.

Viewpoint:
Minimal representation ($\Leftarrow$ group)
$\approx$ Maximal symmetries ($\Leftarrow$ rep. space)

Indefinite orthogonal group $O(p+1, q+1)$

Throughout this talk, $p, q \geq 1$, $p+q$: even > 2

$$G = O(p+1, q+1)$$
$$= \{g \in GL(p+q+2, \mathbb{R}) : {}^t g \begin{pmatrix} I_{p+1} & O \\ O & -I_{q+1} \end{pmatrix} g = \begin{pmatrix} I_{p+1} & O \\ O & -I_{q+1} \end{pmatrix}$$

$\cdots$ real simple Lie group of type D

Minimal representation of $G = O(p+1, q+1)$

- $q = 1$
  highest weight module $\oplus$ lowest weight module
  - the bound states of the Hydrogen atom
- $p = q$
  spherical case
  - $p = q = 3$ case: Kostant (1990)
- p, q: general
  non-highest, non-spherical
  - algebraic construction (e.g. dual pair)
    (Binegar–Zierau, Howe–Tan, Huang–Zhu)
  - construction by conformal geometry (K–Ørsted)
  - L^2 construction (K–Ørsted, K–Mano)

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal model Theorem B	Clear	$\boxed{?}$
v.s.		
2. L^2 model (Schrödinger model) Theorem D	$\boxed{?}$	Clear

Clear Picture $\cdots$ advantage of the model

No single model of minimal models has clear pictures for both group actions and Hilbert structures

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal model Theorem B	Clear	Theorem C
v.s.		
2. L^2 model (Schrödinger model) Theorem D	Theorem E	Clear

Clear Picture $\cdots$ advantage of the model

3. Deformation of Fourier transforms (Theorems F, G, H)
(interpolation, special functions, Dunkl operators)

§1 Conformal construction of minimal reps.

Idea: Composition of holomorphic functions

holomorphic ∘ holomorphic = holomorphic

$\Downarrow$ taking real parts

$\boxed{\text{harmonic} \circ \text{conformal} = \text{harmonic}}$ on $\mathbb{C} \simeq \mathbb{R}^2$

make sense for general Riemannian manifolds.

But $\boxed{\text{harmonic} \circ \text{conformal} \neq \text{harmonic}}$ in general

$\Longrightarrow$ Try to modify the definition!

$\mathrm{Conf}(X, g) \supset \mathrm{Isom}(X, g)$

(X, g) pseudo-Riemannian manifold

$\varphi \in \mathrm{Diffeo}(X)$

<u>Def.</u>

φ is isometry $\iff \varphi^* g = g$

φ is conformal $\iff$ $^\exists$positive function $C_\varphi \in C^\infty(X)$ s.t.

$$\varphi^* g = C_\varphi^2 \, g$$

C_φ : conformal factor

$$\underset{}{\mathrm{Diffeo}(X)} \supset \underset{\text{Conformal group}}{\mathrm{Conf}(X, g)} \supset \underset{\text{isometry group}}{\mathrm{Isom}(X, g)}$$

Harmonic $\circ$ conformal $\neq$ harmonic

Modification

$\varphi \in \mathrm{Conf}(X^n, g)$, $\varphi^* g = C_\varphi^2 g$

- pull-back $\rightsquigarrow$ twisted pull-back

  $f \circ \varphi \quad \rightsquigarrow \quad \underbrace{C_\varphi^{-\frac{n-2}{2}}}_{\text{conformal factor}} f \circ \varphi$

- $\mathcal{S}ol(\Delta_X) = \{f \in C^\infty(X) : \Delta_X f = 0\}$ (harmonic functions)

  $\rightsquigarrow \mathcal{S}ol(\widetilde{\Delta_X}) = \{f \in C^\infty(X) : \widetilde{\Delta_X} f = 0\}$

  $$\underbrace{\widetilde{\Delta_X}}_{\text{Yamabe operator}} := \underbrace{\Delta_X}_{\text{Laplacian}} + \frac{n-2}{4(n-1)} \underbrace{\kappa}_{\text{scalar curvature}}$$

Distinguished rep. of conformal groups

harmonic $\circ$ conformal $\doteqdot$ harmonic

$\Downarrow$ Modification

Theorem A ([K–Ørsted 03]) (X^n, g): pseudo- Riemannian mfd

$\Longrightarrow \mathrm{Conf}(X, g)$ acts on $\mathcal{S}ol(\widetilde{\Delta_X})$ by $f \mapsto C_\varphi^{-\frac{n-2}{2}} f \circ \varphi$

Point $\widetilde{\Delta_X} = \Delta_X + \frac{n-2}{4(n-1)}\kappa$

$\widetilde{\Delta_X}$ is not invariant by $\mathrm{Conf}(X, g)$.

But $\mathcal{S}ol(\widetilde{\Delta_X})$ is invariant by $\mathrm{Conf}(X, g)$.

$$\mathrm{Diffeo}(X) \supset \underbrace{\mathrm{Conf}(X, g)}_{\text{Conformal group}} \supset \underbrace{\mathrm{Isom}(X, g)}_{\text{isometry group}}$$

Application of Theorem A

$$(X,g) := (S^p \times S^q,\ \underbrace{+\cdots+}_{p}\ \underbrace{-\cdots-}_{q})$$

Theorem B ([7, Part I]) $\widetilde{\Delta_X} = \Delta_{S^p} - \Delta_{S^q} + \text{const.}$

0) $\mathrm{Conf}(X,g) \simeq O(p+1,q+1)$

1) $\mathcal{S}ol(\widetilde{\Delta_X}) \neq \{0\} \Longleftrightarrow p+q$ even

2) If $p+q$ is even and > 2, then
$\mathrm{Conf}(X,g) \curvearrowright \mathcal{S}ol(\widetilde{\Delta_X})$ is irreducible,
and for $p+q > 6$ it is a minimal rep of $O(p+1,q+1)$.

↑

$^{\exists}$a $\mathrm{Conf}(X,g)$-invariant inner product, and
take the Hilbert completion

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction		
Theorem B	Clear	$\boxed{?}$

v.s.

Clear ··· advantage of the model

Flat model

Stereographic projection

$$S^n \to \mathbb{R}^n \cup \{\infty\} \qquad \text{conformal map}$$

More generally

$$\underset{+\cdots+}{S^p} \times \underset{-\cdots-}{S^q} \hookleftarrow \underset{ds^2=dx_1^2+\cdots+dx_p^2-dx_{p+1}^2-\cdots-dx_{p+q}^2}{\mathbb{R}^{p+q}} \qquad \text{conformal embedding}$$

Functoriality of Theorem A

$$\begin{tikzcd}
\mathcal{S}ol(\widetilde{\Delta}_{S^p\times S^q}) \arrow[r, phantom, "\subset"] & \mathcal{S}ol(\widetilde{\Delta}_{\mathbb{R}^{p,q}}) \\
\circlearrowleft & \circlearrowleft \\
\mathrm{Conf}(S^p \times S^q) & \mathrm{Conf}(\mathbb{R}^{p,q}) \arrow[l, hook']
\end{tikzcd}$$

Conservative quantity for ultra-hyperbolic eqn.

$$\mathbb{R}^{p,q} = \mathbb{R}^{p+q},\ ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2$$

$$\widetilde{\Delta}_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \Box_{p,q}$$

Unitarization of subrep (representation theory)

$\Longleftrightarrow$

Conservative quantity (differential eqn)

Conservative quantity for ultra-hyperbolic eqn.

$$\mathbb{R}^{p,q} = \mathbb{R}^{p+q},\ ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2$$

$$\widetilde{\Delta}_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \Box_{p,q}$$

<ins>Problem</ins> Find an 'intrinsic' inner product on (a 'large' subspace of) $Sol(\Box_{p,q})$ if exists.

Conservative quantity for ultra-hyperbolic eqn.

$$\mathbb{R}^{p,q} = \mathbb{R}^{p+q},\ ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2$$

$$\widetilde{\Delta}_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \Box_{p,q}$$

$q = 1$ wave operator

<ins>energy</ins> $\cdots$ conservative quantity for wave equations
w.r.t. time translation $\mathbb{R}$

$$\Downarrow$$

$\boxed{\quad ? \quad}$ $\cdots$ conservative quantity for ultra-hyperbolic eqs
w.r.t. conformal group $O(p+1, q+1)$

Conservative quantity for $\Box_{p,q} f = 0$

Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in \mathcal{S}ol(\Box_{p,q})$

$$(f, f) := \int_\alpha Q_\alpha f \quad \text{(to be defined soon)} \qquad \cdots\cdots ①$$

Theorem C ([7, Part III]$+\varepsilon$)
1) ① is independent of hyperplane α.
2) ① gives the unique inner product (up to scalar) which is invariant under $O(p+1, q+1)$.

$\cancel{O(p,q)}$ $\curvearrowright$ $\mathbb{R}^{p,q}$ ~~(linear)~~

$O(p+1, q+1)$ (Möbius transform)

Parametrization of non-characteristic hyperplane

$$\mathbb{R}^{p,q} = (\mathbb{R}^{p+q}, ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2)$$

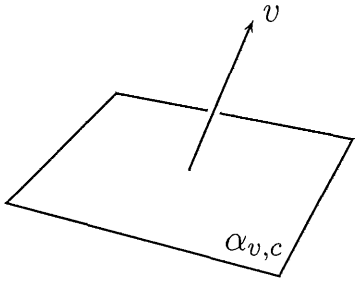
Fix $v \in \mathbb{R}^{p,q}$ s.t. $(v, v)_{\mathbb{R}^{p,q}} = \pm 1$

$c \in \mathbb{R}$

$\Downarrow$

$\mathbb{R}^{p,q} \supset \alpha \equiv \alpha_{v,c} := \{x \in \mathbb{R}^{p+q} : (x, v)_{\mathbb{R}^{p,q}} = c\}$

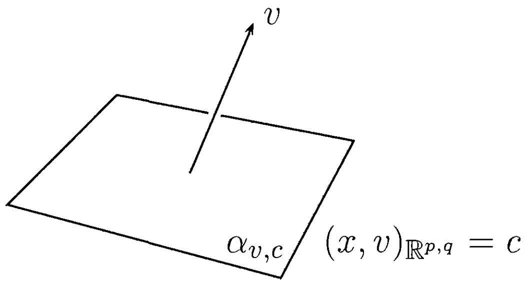
non-characteristic hyperplane



‘Intrinsic’ inner product

For $\alpha = \alpha_{v,c}$, $f \in C^\infty(\mathbb{R}^{p,q})$ with some decay at ∞
Point: $f = f_+ + f_-$ (idea: Sato’s hyperfunction)
$f'_\pm \cdots$ normal derivative of $f_\pm$ w.r.t. v

$$Q_\alpha f := \frac{1}{i}\left(f_+\overline{f'_+} - f_-\overline{f'_-}\right)$$



Conservative quantity for $\Box_{p,q} f = 0$

Fix $\alpha = \alpha_{v,c} \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in \mathcal{S}ol(\Box_{p,q})$

$$(f,f) := \int_\alpha Q_\alpha f \qquad \cdots\cdots ①$$

Theorem C

1) ① is independent of hyperplane α.
2) ① gives the unique inner product (up to scalar) which is invariant under $O(p+1,q+1)$.

Theorem C is non-trivial even for $q = 1$ (wave equation)
In space-time $\mathbb{R}^{p+1} = \mathbb{R}^p_x \times \mathbb{R}_t$,
average in space (i.e. time $t =$ constant)
$=$ average in (any hyperplane in space) $\times \mathbb{R}_t$ (time)

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction Theorems A, B	Clear	?
v.s.		
2. ?	?	Clear

Clear · · · advantage of the model

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction Theorems A, B	Clear	conservative quantity Theorem C
v.s.		
2. ?	?	Clear

Clear · · · advantage of the model

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction Theorems A, B	Clear	$\boxed{\text{conservative quantity Theorem C}}$
v.s.		
2. L^2 construction (Schrödinger model) Theorem D	$\boxed{?}$	Clear

Clear $\cdots$ advantage of the model

Conformal model $\Longrightarrow$ L^2-model

$$\Box_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

$$\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$$

Conformal model $\Longrightarrow$ L^2-model

$$\Box_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

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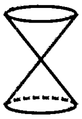
$=$  (figure for $(p, q) = (2, 1)$)

Conformal model $\Longrightarrow$ L^2-model

$$\Box_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

$$\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$$

$$\boxed{\Box_{p,q} f = 0 \underset{\text{Fourier trans.}}{\Longrightarrow} \operatorname{Supp} \mathcal{F}f \subset \Xi}$$

$$\begin{array}{cccc} \mathcal{F}: & \mathcal{S}'(\mathbb{R}^{p,q}) & \xrightarrow{\sim} & \mathcal{S}'(\mathbb{R}^{p,q}) \\ & \cup & & \cup \end{array}$$

$$\boxed{\underline{\text{Theorem D}}\ (\underline{[7,\ \text{Part III}]}) \quad \overline{\mathcal{S}ol(\Box_{p,q})} \xrightarrow{\sim} L^2(\Xi)}$$

conformal model $\qquad$ L^2-model

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction Theorems A, B	Clear	$\boxed{\text{conservative quantity}}$
v.s.		
2. L^2 construction (Schrödinger model) Theorem D	$\boxed{\ ?\ }$	Clear

Clear $\cdots$ advantage of the model

§2 L^2-model of minimal reps.

$$\boxed{\underline{\text{Theorem D}}\quad p+q>2,\ \text{even.}\quad \overline{Sol(\Box_{p,q})} \xrightarrow{\ \sim\ } L^2(\Xi)}$$

conformal model $\qquad$ L^2-model

$$G = O(p+1,q+1) \overset{\text{minimal rep.}}{\curvearrowright} L^2(\Xi) \qquad \text{unitary rep.}$$

$$\boxed{\dim \Xi = p+q-1 \implies \Xi \text{ is too small to be acted by } G.}$$

$$\begin{array}{lcl} & \not\curvearrowright & \Xi \subset \mathbb{R}^{p,q} \subset \mathbb{R}^{p+1,q+1} \\ O(p+1,q+1) & & \\ & \curvearrowright & L^2(\Xi) \end{array}$$

Ξ as Lagrangian in $\mathcal{O}_{\min}$

$\mathfrak{g}^* \supset \mathcal{O}_{\min} = \mathrm{Ad}^*(G)\lambda$ minimal nilp. orbit
$\updownarrow$? "geometric quantization"
$\widehat{G} \ni \pi$ minimal rep of G

Assume $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ parabolic s.t. $\lambda|_{\mathfrak{p}} \equiv 0$
$\Longrightarrow \Xi := \mathfrak{n} \cap \mathcal{O}_{\min}$ is isotropic in $\mathcal{O}_{\min}$

<u>Ex</u> $G = Sp(n, \mathbb{R})^{\sim}$, $\mathfrak{p}$ = Siegel parabolic
$\Rightarrow \mathcal{O}_{\min} \supset \Xi$ Lagrangian
$\mathbb{R}^n \setminus \{0\} \xrightarrow{\text{double cover}} \Xi, \quad x \mapsto x\,{}^t x$
$G \overset{\pi}{\curvearrowright} L^2(\mathbb{R}^n)_{\text{even}} \overset{\sim}{\leftarrow} L^2(\Xi)$
Schrödinger model of Segal–Shale–Weil rep.

Ξ as Lagrangian in $\mathcal{O}_{\min}$

$\mathfrak{g}^* \supset \mathcal{O}_{\min} = \mathrm{Ad}^*(G)\lambda$ minimal nilp. orbit
$\updownarrow$? "geometric quantization"
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Assume $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ parabolic s.t. $\lambda|_{\mathfrak{p}} \equiv 0$
$\Longrightarrow \Xi := \mathfrak{n} \cap \mathcal{O}_{\min}$ is isotropic in $\mathcal{O}_{\min}$

<u>Ex</u> $G = O(p+1, q+1)$, $\mathfrak{p} = \mathfrak{conf}(S^p \times S^q)$
$\Rightarrow \mathcal{O}_{\min} \supset \Xi$ Lagrangian

$G \curvearrowright L^2(\Xi)$
L^2-model of minimal rep. (Theorem D)

Inversion element

$$G = PGL(2,\mathbb{C}) \underset{\text{Möbius transform}}{\curvearrowright} \mathbb{P}^1\mathbb{C} \simeq \mathbb{C} \cup \{\infty\}$$

$$\doteqdot O(3,1) \qquad\qquad \doteqdot \mathbb{R}^{2,0}$$

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times,\ b \in \mathbb{C} \right\} \qquad z \mapsto az + b$$

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad z \mapsto -\tfrac{1}{z} \quad \text{(inversion)}$$

G is generated by P and w.

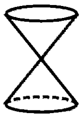
$$G = O(p+1, q+1) \underset{\text{Möbius transform}}{\curvearrowright} \mathbb{R}^{p,q}$$

$$P = \{(A,b) : A \in O(p,q) \cdot \mathbb{R}^\times,\ b \in \mathbb{R}^{p+q}\} \quad x \mapsto Ax + b$$

$$w = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix} : (x', x'') \mapsto \frac{4}{|x'|^2 - |x''|^2}(-x', x'') \quad \text{(inversion)}$$

New Fourier transform $\mathcal{F}_\Xi$ on Ξ

$$\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$$

=  (figure for $(p,q) = (2,1)$)

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$\mathcal{F}_\Xi$ on $\Xi =$ 

Problem Define new Fourier trans. $\mathcal{F}_\Xi$.

‘Fourier transform’ $\mathcal{F}_\Xi$ on Ξ

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$$\mathcal{F}^4 = \mathrm{id}$$

$\mathcal{F}_\Xi$ on $\Xi =$ 

$$\mathcal{F}_\Xi^2 = \mathrm{id}$$

‘Fourier transform’ $\mathcal{F}_\Xi$ on Ξ

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$$Q_j \mapsto -P_j$$
$$P_j \mapsto Q_j$$

$\mathcal{F}_\Xi$ on $\Xi =$ 

$$Q_j \mapsto R_j$$
$$R_j \mapsto Q_j$$

$Q_j = x_j$ (multiplication by coordinates function)

$$P_j = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_j}$$

$R_j = {}^\exists$second order differential op. on Ξ

Rediscover Bargmann–Todorov’s operators

‘Fourier transform’ $\mathcal{F}_\Xi$ on Ξ

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$$\begin{array}{ccc} Q_j & \mapsto & -P_j \\ P_j & \mapsto & Q_j \end{array}$$

$\mathcal{F}_\Xi$ on $\Xi =$ 

$$\begin{array}{ccc} Q_j & \mapsto & R_j \\ R_j & \mapsto & Q_j \end{array}$$

$Q_j = x_j$ (multiplication by coordinates function)

$$P_j = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_j}$$

$R_j = {}^{\exists}$second order differential op. on Ξ

Notice
$$\left.\begin{array}{l} {Q_1}^2 + \cdots + {Q_p}^2 - {Q_{p+1}}^2 - \cdots - {Q_{p+q}}^2 = 0 \\ {R_1}^2 + \cdots + {R_p}^2 - {R_{p+1}}^2 - \cdots - {R_{p+q}}^2 = 0 \end{array}\right\} \text{on } \Xi$$

Unitary inversion operator $\mathcal{F}_\Xi$

$p+q$: even > 2

$G = O(p+1, q+1) \curvearrowright L^2(\Xi)$ minimal rep.

w-action $\cdots$ $\mathcal{F}_\Xi$ (unitary inversion operator)

Problem Find the unitary operaotr $\mathcal{F}_\Xi$ explicitly.

Cf. Euclidean case $\varphi(t) = e^{-it}$ (one variable)

$\mathcal{F}_{\mathbb{R}^N} f(x) = c\int_{\mathbb{R}^N} \varphi(\langle x, y\rangle) f(y) dy$

Thm E (K–Mano, to appear in Memoirs AMS)

$$(\mathcal{F}_\Xi f)(x) = c\int_\Xi \Phi^{\varepsilon(p,q)}_{\frac{1}{2}(p+q-4)}(\langle x, y\rangle) f(y) dy$$

$\mathcal{F}_{\mathbb{R}^N}$ v.s. $\mathcal{F}_{\Xi}$

On $\mathbb{R}^N$

$$\boxed{\begin{array}{l} (\mathcal{F}_{\mathbb{R}^N} f)(x) = c \int_{\mathbb{R}^N} \varphi(\langle x, y\rangle) f(y) dy \\ \varphi(t) = e^{-it} \text{ satisfies} \\ \left(\frac{d}{dt} + i\right) \varphi(t) = 0 \end{array}}$$

On Ξ $(\subset \mathbb{R}^{p,q})$

$$\boxed{\begin{array}{l} (\mathcal{F}_{\Xi} f)(x) = c \int_{\Xi} \Phi(\langle x, y\rangle) f(y) dy \\ \Phi(t) \text{ satisfies} \\ \left(\left(t\frac{d}{dt}\right)^2 + \frac{1}{2}(p+q-4)t\frac{d}{dt} + 2t\right) \Phi(t) = 0 \end{array}}$$

Mellin–Barnes type integral

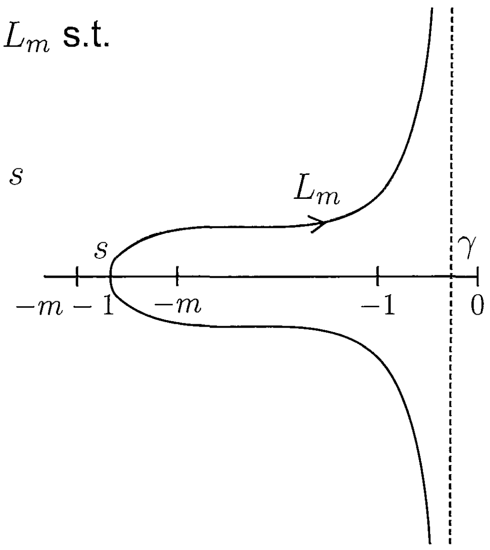
Idea: Apply Mellin–Barnes type integral to distributions.

Fix $m \in \mathbb{N}$. Take a contour L_m s.t.

1) L_m starts at $\gamma - i\infty$
2) passes the real axis at s
3) ends at $\gamma + i\infty$

where

$$-m-1 < s < -m$$
$$-1 < \gamma < 0$$



Explicit formula of $\mathcal{F}_\Xi$ on Ξ

Theorem E ([5]) Suppose $p+q$: even > 2

$$(\mathcal{F}_\Xi f)(x) = c\int_\Xi \Phi^{\varepsilon(p,q)}_{\frac{1}{2}(p+q-4)}(\langle x,y\rangle)f(y)dy$$

Here, $\varepsilon(p,q) = \begin{cases} 0 & \text{if } \min(p,q)=1, \\ 1 & \text{if } p,q>1 \text{ are both odd,} \\ 2 & \text{if } p,q>1 \text{ are both even.}\end{cases}$

$$\Phi^\varepsilon_m(t) = \begin{cases} \displaystyle\int_{L_0} \frac{\Gamma(-\lambda)}{\Gamma(\lambda+1+m)}(2t)^\lambda_+ d\lambda & (\varepsilon=0) \\ \displaystyle\int_{L_m} \frac{\Gamma(-\lambda)}{\Gamma(\lambda+1+m)}(2t)^\lambda_+ d\lambda & (\varepsilon=1) \\ \displaystyle\int_{L_m} \frac{\Gamma(-\lambda)}{\Gamma(\lambda+1+m)}\left(\frac{(2t)^\lambda_+}{\tan(\pi\lambda)}+\frac{(2t)^\lambda_-}{\sin(\pi\lambda)}\right) d\lambda & (\varepsilon=2)\end{cases}$$

Regularity of $\Phi^\varepsilon_m(t)$

Cf. Euclidean Fourier transform $e^{-it} \in \mathcal{A}(\mathbb{R}) \cap L^1_{\mathrm{loc}}(\mathbb{R}) \cap \cdots$

Recall two distributions on $\mathbb{R}$

$\delta(t)$: Dirac's delta function

t^{-1}: Cauchy's principal value

$$= \lim_{s\to 0}\left(\int_{-\infty}^{-s}+\int_s^\infty\right)\langle\frac{1}{t},\cdot\rangle dt$$

these are not in $L^1_{\mathrm{loc}}(\mathbb{R})$

Regularity of $\Phi^{\varepsilon}_{m}(t)$

Cf. Euclidean Fourier transform $e^{-it} \in \mathcal{A}(\mathbb{R}) \cap L^1_{\mathrm{loc}}(\mathbb{R}) \cap \cdots$

<u>Prop.</u> ([<u>K–Mano</u>]) We have the identities $\mod L^1_{\mathrm{loc}}(\mathbb{R})$

$$\Phi^{\varepsilon}_{m}(t) \equiv \begin{cases} 0 & (\varepsilon = 0) \\ -\pi i \displaystyle\sum_{l=0}^{m-1} \frac{(-1)^l}{2^l (m-l-1)!} \delta^{(l)}(t) & (\varepsilon = 1) \\ -i \displaystyle\sum_{l=0}^{m-1} \frac{l!}{2^l (m-l-1)!} t^{-l-1} & (\varepsilon = 2) \end{cases}$$

<u>Cor.</u> $\mathcal{F}_{\Xi}$ has a locally integrable kernel if and only if G is $O(p+1,2)$, $O(2,q+1)$, or $O(3,3)$ $(\doteqdot SL(4,\mathbb{R}))$.

Bessel functions

$$J_\nu(z) = \left(\frac{z}{2}\right)^{\nu} \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{z}{2}\right)^{2j}}{j!\,\Gamma(j+\nu+1)}$$

$$I_\nu(z) := e^{-\frac{\sqrt{-1}\nu\pi}{2}} J_\nu\left(e^{\frac{\sqrt{-1}\pi}{2}} z\right)$$

$$Y_\nu(z) := \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi} \qquad \text{(second kind)}$$

$$K_\nu(z) := \frac{\pi}{2\sin\nu\pi}\left(I_{-\nu}(z) - I_\nu(z)\right) \qquad \text{(third kind)}$$

Bessel distribution

Prop. ([4]) $\Phi_m^{\varepsilon}(t)$ solves the differential equation

$$(\theta^2 + m\theta + 2t)u = 0$$

where $\theta = t\frac{d}{dt}$.

Explicit forms

$$\Phi_m^0(t) = 2\pi i (2t)_+^{-\frac{m}{2}} J_m(2\sqrt{2t_+})$$

$$\Phi_m^1(t) = \Phi_m^0(t) - \pi i \sum_{l=0}^{m-1} \frac{(-1)^l}{2^l(m-l-1)!}\delta^{(l)}(t)$$

$$\Phi_m^2(t) = 2\pi i (2t)_+^{-\frac{m}{2}} Y_m(2\sqrt{2t_+}) + 4(-1)^{m+1} i\,(2t)_-^{-\frac{m}{2}} K_m(2\sqrt{2t_-})$$

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction Theorems A, B	Clear	conservative quantity
v.s.		
2. L^2 construction (Schrödinger model) Theorem D	‘Fourier transform’ $\mathcal{F}_\Xi$	Clear

Clear $\cdots$ advantage of the model

3. Deformation of Fourier transforms (Theorems F, G, H)

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction Theorems A, B	Clear	Theorem C
v.s.		
2. L^2 construction (Schrödinger model) Theorem D	Theorem E	Clear

Clear $\cdots$ advantage of the model

3. Deformation of Fourier transforms (Theorems F, G, H)

Application to special functions

Minimal reps ($\Leftarrow$ group)
$\approx$ Maximal symmetries ($\Leftarrow$ space)

$\Rightarrow$ 'Special functions', 'orthogonal polynomials' associated to 4th order differential eqn [3a, 3b, 3c]

with J.Hilgert, G.Mano, and J.Moellers

with 4 parameters

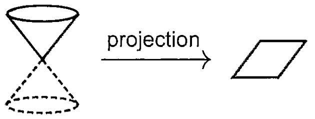
$$(\underbrace{p, q}_{\text{dimension}} ; \underbrace{l, m}_{\text{branching laws (multiplicity-free)}})$$

Special case $q = 1$: Laguerre polynomials $4 = 2 \times 2$

Interpolation of Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

$\mathcal{F}_\Xi$ $\cdots$ 'Fourier transform' on $\Xi \subset \mathbb{R}^{p,q}$
$\mathcal{F}_{\mathbb{R}^N}$ $\cdots$ Fourier transform on $\mathbb{R}^N$

Assume $q = 1$. Set $p = N$.



$\mathbb{R}^{N,1} \supset \Xi =$ $\xrightarrow{\text{projection}}$ $= \mathbb{R}^N$

$\mathcal{F}_\Xi$ $\qquad$ $\mathcal{F}_{\mathbb{R}^N}$

$O(N+1,2)$ $\qquad$ $Mp(N,\mathbb{R})$

Interpolation of Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

$\mathcal{F}_\Xi$ $\cdots$ 'Fourier transform' on $\Xi \subset \mathbb{R}^{p,q}$
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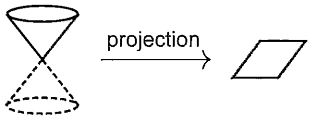


$\mathbb{R}^{N,1} \supset \Xi =$ $\xrightarrow{\text{projection}}$ $= \mathbb{R}^N$

$\mathcal{F}_\Xi \overset{\text{interpolate}}{\cdots\cdots\cdots} \mathcal{F}_{\mathbb{R}^N}$

$a = 1$ $\qquad$ $a = 2$

(k, a)-deformation of $\exp\frac{t}{2}(\Delta - |x|^2)$

Fourier transform

$$\boxed{\mathcal{F}_{\mathbb{R}^N} = \underbrace{c}_{\substack{\text{phase factor} \\ = e^{\frac{\pi i N}{4}}}} \exp\Big(\frac{\pi i}{4}\overbrace{(\underbrace{\Delta}_{\text{Laplacian}} - |x|^2)}^{\text{self-adjoint op. on } L^2(\mathbb{R}^N)}\Big)}$$

Hermite semigroup

$$\boxed{I(t) := \exp\frac{t}{2}(\Delta - |x|^2)}$$

Mehler kernel using $\exp(-x^2)$

(k, a)-deformation of $\exp\frac{t}{2}(\Delta - |x|^2)$

Hankel-type transform on Ξ

$$\boxed{\mathcal{F}_{\Xi} = \underbrace{c}_{\substack{\text{phase factor} \\ = e^{\frac{\pi i (N-1)}{2}}}} \exp\Big(\frac{\pi i}{2}\overbrace{(\underbrace{|x|\Delta}_{\text{Laplacian}} - |x|)}^{\text{self-adjoint op. on } L^2(\mathbb{R}^N, \frac{dx}{|x|})}\Big)}$$

"Laguerre semigroup" ([K–Mano], 2007)

$$\boxed{\mathcal{I}(t) := \exp t(|x|\Delta - |x|)} \quad \operatorname{Re} t > 0$$

closed formula using Bessel function

(k,a)-deformation of $\exp\frac{t}{2}(\Delta - |x|^2)$

(k,a)-generalized Fourier transform

$$\boxed{\mathcal{F}_{k,a} = \underbrace{c}_{\substack{\text{phase factor}\\ = e^{i\frac{\pi(N+2\langle k\rangle+a-2)}{2a}}}} \exp\Big(\frac{\pi i}{2a}\big(\overbrace{|x|^{2-a}\underbrace{\Delta_k}_{\text{Dunkl Laplacian}} - |x|^a}^{\text{self-adjoint op. on } L^2(\mathbb{R}^N,\, \vartheta_{k,a}(x)dx)}\big)\Big)}$$

(k,a)-deformation of Hermite semigroup ([BKO])

$$\boxed{\mathcal{I}_{k,a}(t) := \exp\frac{t}{a}(|x|^{2-a}\Delta_k - |x|^a)} \quad \mathrm{Re}\, t > 0$$

k: multiplicity on root system $\mathcal{R}$, $a > 0$

(k,a)-deformation of Hermite semigp

$k = (k_\alpha)$: multiplicity of root system $\mathcal{R}$ in $\mathbb{R}^N$

$\mathcal{H}_{k,a} := L^2(\mathbb{R}^N,\ |x|^{a-2} \prod_{\alpha\in\mathcal{R}} |\langle x, \alpha\rangle|^{k_\alpha} dx)$

Thm F ([with Ben Saïd and Ørsted])
Assume $a > 0$ and $a + \sum k_\alpha + N - 2 > 0$.
$\mathcal{I}_{k,a}(t) := \exp\frac{t}{a}(|x|^{2-a}\Delta_k - |x|^a)$ is a holomorphic semigp on $\mathcal{H}_{k,a}$ for $\mathrm{Re}\, t > 0$.

Point: The unitary rep on $\mathcal{H}_{k,a}$ is $\widetilde{SL(2,\mathbb{R})}$-admissible
(i.e. discretely decomposable and finite multiplicities)

$\Longrightarrow {}^\forall$ Spectrum of $|x|^{2-a}\Delta_k - |x|^a$ is discrete and negative

(k,a)-deformation of Hermite semigp

$k=(k_\alpha)$: multiplicity of root system $\mathcal{R}$ in $\mathbb{R}^N$

$\mathcal{H}_{k,a} := L^2(\mathbb{R}^N,\ |x|^{a-2}\prod_{\alpha\in\mathcal{R}}|\langle x,\alpha\rangle|^{k_\alpha}dx)$

Thm F ([with Ben Saïd and Ørsted])

Assume $a>0$ and $a+\sum k_\alpha+N-2>0$.

$\mathcal{I}_{k,a}(t):=\exp\frac{t}{a}(|x|^{2-a}\Delta_k-|x|^a)$ is a holomorphic semigp on $\mathcal{H}_{k,a}$ for $\operatorname{Re}t>0$.

$\mathcal{I}_{k,a}(t_1)\circ\mathcal{I}_{k,a}(t_2)=\mathcal{I}_{k,a}(t_1+t_2)$ for $\operatorname{Re}t_1,t_2\geq 0$

$(\mathcal{I}_{k,a}(t)f,g)$ is holomorphic for $\operatorname{Re}t>0$, for ${}^\forall f,{}^\forall g$

(k,a)-deformation of Hermite semigp

$k=(k_\alpha)$: multiplicity of root system $\mathcal{R}$ in $\mathbb{R}^N$

$\mathcal{H}_{k,a} := L^2(\mathbb{R}^N,\ |x|^{a-2}\prod_{\alpha\in\mathcal{R}}|\langle x,\alpha\rangle|^{k_\alpha}dx)$

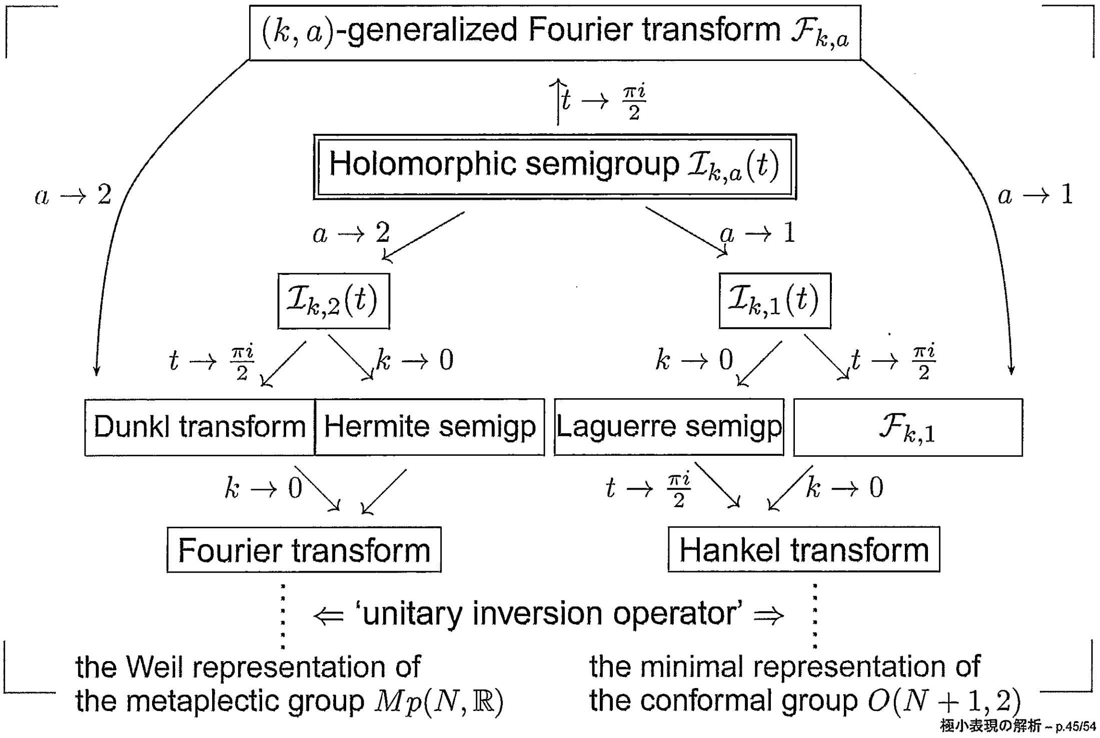
Thm F ([with Ben Saïd and Ørsted])

Assume $a>0$ and $a+\sum k_\alpha+N-2>0$.

$\mathcal{I}_{k,a}(t):=\exp\frac{t}{a}(|x|^{2-a}\Delta_k-|x|^a)$ is a holomorphic semigp on $\mathcal{H}_{k,a}$ for $\operatorname{Re}t>0$.

$$\mathcal{F}_{k,a}:=\underbrace{c}_{\substack{\text{phase factor}\\ e^{i\frac{\pi(N+2\langle k\rangle+a-2)}{2a}}}}\mathcal{I}_{k,a}\left(\frac{\pi i}{2}\right)$$

Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$


(k,a)-generalized Fourier transform $\mathcal{F}_{k,a}$

$t \to \frac{\pi i}{2}$

Holomorphic semigroup $\mathcal{I}_{k,a}(t)$

$a \to 2$ $\quad$ $a \to 1$

$a \to 2$ $\quad$ $a \to 1$

$\mathcal{I}_{k,2}(t)$ $\quad$ $\mathcal{I}_{k,1}(t)$

$t \to \frac{\pi i}{2}$ $\quad$ $k \to 0$ $\quad$ $k \to 0$ $\quad$ $t \to \frac{\pi i}{2}$

Dunkl transform | Hermite semigp | Laguerre semigp | $\mathcal{F}_{k,1}$

$k \to 0$ $\quad$ $t \to \frac{\pi i}{2}$ $\quad$ $k \to 0$

Fourier transform $\quad$ Hankel transform

⇐ ‘unitary inversion operator’ ⇒

the Weil representation of
the metaplectic group $Mp(N,\mathbb{R})$

the minimal representation of
the conformal group $O(N+1,2)$

Generalized Fourier transform $\mathcal{F}_{k,a}$

$$\mathcal{F}_{k,a} = c\mathcal{I}_{k,a}(\frac{\pi i}{2}) = c\exp\left(\frac{\pi i}{2a}(|x|^{2-a}\Delta_k - |x|^a)\right)$$

Thm G ([4]) 1) $\mathcal{F}_{k,a}$ is a unitary operator

2) $\mathcal{F}_{0,2}$ = Fourier transform on $\mathbb{R}^N$

$F_{k,a}$ = Dunkl transform on $\mathbb{R}^N$

$\mathcal{F}_{0,1}$ = Hankel-type transform on L^2(⧗)

3) $\mathcal{F}_{k,a}$ is of finite order $\Longleftrightarrow a \in \mathbb{Q}$

4) $\mathcal{F}_{k,a}$ intertwines $|x|^a$ and $-|x|^{2-a}\Delta_k$

$\Longrightarrow$ generalization of classical identities such as Hecke identity, Bochner identity, Parseval–Plancherel formulas, Weber’s second exponential integral, etc.

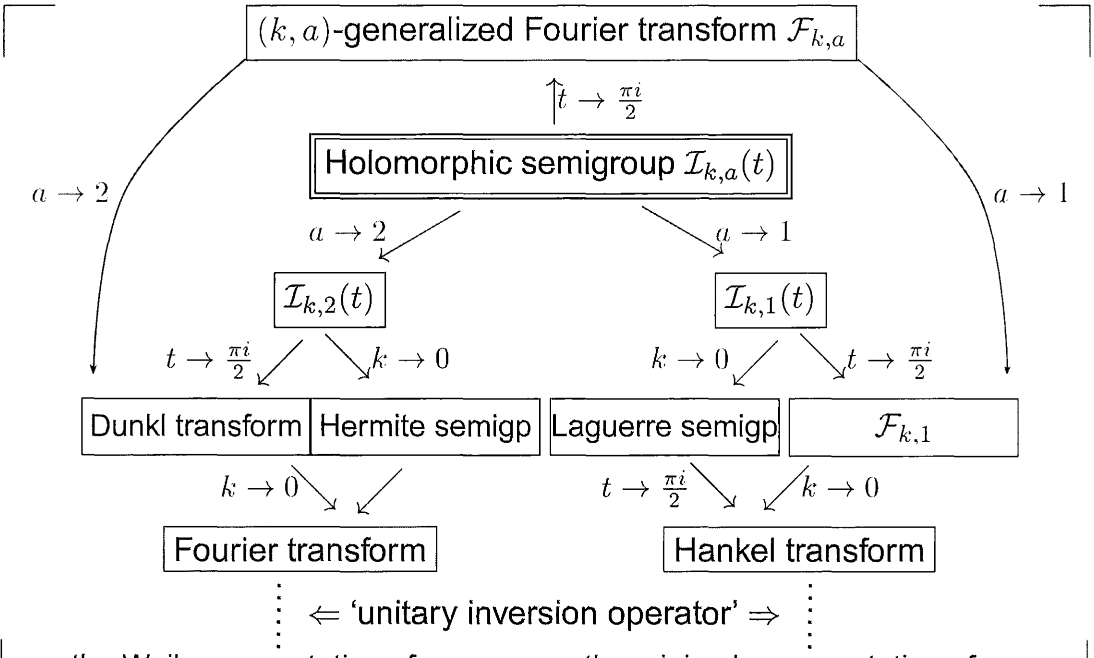
Heisenberg-type inequality

Thm H ([2]) (Heisenberg inequality)

$$\| |x|^{\frac{a}{2}} f(x) \|_k \; \| |\xi|^{\frac{a}{2}} (\mathcal{F}_{k,a} f)(\xi) \|_k \geq \frac{2\langle k \rangle + N + a - 2}{2} \| f(x) \|_k^2$$

$k \equiv 0$, $a = 2$	··· Weyl–Pauli–Heisenberg inequality for Fourier transform $\mathcal{F}_{\mathbb{R}^N}$
k: general, $a = 2$	··· Heisenberg inequality for Dunkl transform $\mathcal{D}_k$ (Rösler, Shimeno)
$k \equiv 0$, $a = 1$, $N = 1$	··· Heisenberg inequality for Hankel transform

Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$



the Weil representation of the metaplectic group $Mp(N, \mathbb{R})$

the minimal representation of the conformal group $O(N + 1, 2)$

Hidden symmetries in $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$

Coxeter group

$$\begin{tikzcd}
 & & \boxed{O(N+1,2)^{\sim}} \\
\boxed{\mathfrak{C} \times \widetilde{SL(2,\mathbb{R})}} \arrow[r, "k \to 0"] & \boxed{O(N) \times \widetilde{SL(2,\mathbb{R})}} \arrow[ur, "a \to 1"'] \arrow[dr, "a \to 2"] & \\
 & & \boxed{Mp(N,\mathbb{R})}
\end{tikzcd}$$

(k, a : general)

Geometric analysis on minimal reps of $O(p, q)$

[1] Algebraic analysis on minimal reps ··· 28 pp. arXiv:1001.0224

[2] Laguerre semigroup and Dunkl operators ··· 74 pp. arXiv:0907.3749

[3] Special functions associated to a fourth order differential equation ··· 57 pp. arXiv:0907.2608, arXiv:0907.2612, arXiv1003.2699

[4] Generalized Fourier transforms $\mathcal{F}_{k,a}$ ··· C.R.A.S. Paris 2009

[5] Schrödinger model of minimal rep. ··· Memoirs of Amer. Math. Soc. (in press), 171 pp.

[6] Inversion and holomorphic extension ··· R. Howe 60th birthday volume (2007), 65 pp.

[7] Analysis on minimal representations ··· Adv. Math. (2003) I, II, III, 110 pp.

Collaborated with S. Ben Saïd, J. Hilgert, G. Mano, J. Möllers and B. Ørsted