LOG HODGE THEORY AND AN APPLICATION TO CALABI-YAU THREEFOLDS

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ABSTRACT. This is a talk about log Hodge theory constructed in the book:
K. Kato and S. Usui, *Classifying spaces of degenerating polarized Hodge structures*, Ann. of Math. Stud., **169**, Princeton Univ. Press, Princeton, NJ, 2009.
As an application, a generic Torelli theorem of quintic-mirror is proved in section 2.
The next short poem is at the top of this book.

Suugaku wa mugen enten kou kokoro
Koute kogarete harukana tabiji

by Kazuya Kato and Sampei Usui,
which was translated by Luc Illusie as

L'impossible voyage aux points à l'infini
N'a pas fait battre en vain le coeur du géomètre

CONTENTS

§0. INTRODUCTION

This part is a summary of the introduction of the book with K. Kato.

Griffiths defined and studied in [G1] the classifying space D of polarized Hodge structures of fixed weight w and fixed Hodge numbers $(h^{p,q})$, and presented in [G2] a dream to add points at infinity to D. [KU1] is an announcement of our attempt to realize his dream, and this book is its full-detailed version.

In a special case where $w = 1$, $h^{1,0} = h^{0,1} = g$, and other $h^{p,q} = 0$, the classifying space D coincides with Siegel's upper half space $\mathfrak{h}_g$ of degree g. In this case, for a subgroup Γ of $\mathrm{Sp}(g, \mathbf{Z})$ of finite index, toroidal compactifications of $\Gamma\backslash\mathfrak{h}_g$ ([AMRT]) and the Satake-Baily-Borel compactification of $\Gamma\backslash\mathfrak{h}_g$ ([Sa], [BB]) are already constructed, where points at infinity often play more important roles than usual points. For example, in the simplest case $g = 1$, the Taylor expansion of a modular form at the standard cusp (i.e., the class of $\infty \in \mathbf{P}^1(\mathbf{Q})$ modulo Γ) of the compactified modular curve $\Gamma\backslash(\mathfrak{h}\cup\mathbf{P}^1(\mathbf{Q}))$ is called the q-expansion and is very important in the theory of modular forms.

The theory of these compactifications is included in a general theory of compactifications of quotients of symmetric Hermitian domains by the actions of discrete arithmetic groups. However, the classifying space D in general is rarely a symmetric Hermitian domain, and we can not use the general theory of symmetric Hermitian domains when we try to add points at infinity to D. In this book, we overcome this difficulty. We discuss two subjects.

Subject I. Toroidal partial compactifications and moduli of polarized logarithmic Hodge structures.

In this book, for general D, we construct a kind of toroidal partial compactification $\Gamma\backslash D_\Sigma$ of $\Gamma\backslash D$ associated to a fan Σ and a discrete subgroup Γ of $\mathrm{Aut}(D)$ satisfying a certain compatibility with Σ.

In the case $D = \mathfrak{h}_g$, the classes of polarized Hodge structures in $\Gamma\backslash\mathfrak{h}_g$ converge to a point at infinity of $\Gamma\backslash\mathfrak{h}_g$ when the polarized Hodge structures degenerate. As in [Sc], nilpotent orbits appear when polarized Hodge structures degenerate. In our definition of D_Σ for general D, a nilpotent orbit itself is viewed as a point at infinity. In order to do so, we use logarithmic structures introduced by Fontaine and Illusie and developed in [K], [KN]. The theory of nilpotent orbits is regarded as a local aspect of the theory of polarized logarithmic Hodge structures (= PLH). A fundamental observation here is: (a nilpotent orbit) = (a PLH over a logarithmic point).

Our main theorem concerning Subject I is stated roughly as follows.

Theorem. $\Gamma\backslash D_\Sigma$ *is the fine moduli space of "polarized logarithmic Hodge structures" with a "Γ-level structure" whose "local monodromies are in the directions in Σ".*

In the classical case $D = \mathfrak{h}_g$, for a subgroup Γ of $\mathrm{Sp}(g, \mathbf{Z})$ of finite index and for a sufficiently big fan Σ, $\Gamma\backslash D_\Sigma$ is a toroidal compactification of $\Gamma\backslash\mathfrak{h}_g$. Already in this classical case, this theorem gives moduli-theoretic interpretations of the toroidal compactifications of $\Gamma\backslash\mathfrak{h}_g$.

For general D, the space $\Gamma\backslash D_\Sigma$ has a kind of complex structure, but a delicate point is that this space can have locally the shape of "complex analytic space with a slit" (for example, $\mathbf{C}^2$ minus $\{(0, z) \mid z \in \mathbf{C}, z \neq 0\}$), and hence it is often not locally compact. However it is very near to a complex analytic manifold, and we call it a "logarithmic manifold". Infinitesimal calculus is performed on $\Gamma\backslash D_\Sigma$ nicely. These phenomena are first examined in the easiest non-trivial case in [U1].

One motivation of Griffiths for adding points at infinity to D was the hope that the period map $\Delta^* \to \Gamma\backslash D$, associated to a variation of polarized Hodge structure on a punctured disc Δ^*, could be extended over the puncture. By using the above main theorem and the nilpotent orbit theorem of Schmid, we can actually extend the period map to $\Delta \to \Gamma\backslash D_\Sigma$ for some suitable fan Σ.

Subject II. The eight enlargements of D and the fundamental diagram.

In the classical case $D = \mathfrak{h}_g$, there is another compactification $\Gamma\backslash D_{\mathrm{BS}}$ of $\Gamma\backslash\mathfrak{h}_g$, for Γ a subgroup of $\mathrm{Sp}(g, \mathbf{Z})$ of finite index, called Borel-Serre compactification ([BS]). This is a real manifold with corners (like $\mathbf{R}^m_{\geq 0} \times \mathbf{R}^n$ locally).

For general D, by adding to D points at infinity of different kinds, we obtain eight enlargements of D with maps among them which form the following diagram.

Fundamental Diagram.

$$\begin{tikzcd}
 & & D_{\mathrm{SL}(2),\mathrm{val}} \arrow[r, hook] \arrow[d] & D_{\mathrm{BS},\mathrm{val}} \arrow[d] \\
D_{\Sigma,\mathrm{val}} \arrow[d] & D^{\sharp}_{\Sigma,\mathrm{val}} \arrow[l] \arrow[r] \arrow[d] & D_{\mathrm{SL}(2)} & D_{\mathrm{BS}} \\
D_{\Sigma} & D^{\sharp}_{\Sigma} \arrow[l] & &
\end{tikzcd}$$

Note that the space D_Σ, which appeared in Subject I, sits at the left lower end of this diagram. Like nilpotent orbits, SL(2)-orbits also appear in the theory of degenerations of polarized Hodge structures ([Sc], [CKS]). This diagram tells how nilpotent orbits, SL(2)-orbits, and the theory of Borel-Serre are related. The left-hand side of the above diagram has Hodge-theoretic nature, and the right-hand side has the nature of theory of algebraic groups. These are related by the middle map $D^{\sharp}_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ which is a geometric interpretation of SL(2)-orbit theorem of Cattani-Kaplan-Schmid.

The eight spaces D_Σ, $D^{\sharp}_{\Sigma}$, $D_{\mathrm{SL}(2)}$, D_{BS}, $D_{\Sigma,\mathrm{val}}$, $D^{\sharp}_{\Sigma,\mathrm{val}}$, $D_{\mathrm{SL}(2),\mathrm{val}}$, $D_{\mathrm{BS},\mathrm{val}}$ in Fundamental Diagram are defined as the spaces of nilpotent orbits, nilpotent i-orbits, SL(2)-orbits, Borel-Serre orbits, valuative nilpotent orbits, valuative nilpotent i-orbits, valuative SL(2)-orbits, valuative Borel-Serre orbits, respectively. Roughly speaking, $\Gamma\backslash D_\Sigma$ is like an analytic manifold with slits, $D^{\sharp}_{\Sigma}$ and $D_{\mathrm{SL}(2)}$ are like real manifolds with corners and slits, D_{BS} is a real manifold with corners, $\Gamma\backslash D_{\Sigma,\mathrm{val}}$ and $D^{\sharp}_{\Sigma,\mathrm{val}}$ are the projective limits of "blowing-ups" of $\Gamma\backslash D_\Sigma$ and $D^{\sharp}_{\Sigma}$, respectively, associated to rational subdivisions of Σ, $D_{\mathrm{SL}(2),\mathrm{val}}$ and $D_{\mathrm{BS},\mathrm{val}}$ are the projective limits of certain "blowing-ups" of $D_{\mathrm{SL}(2)}$ and D_{BS}, respectively. The maps $\Gamma\backslash D^{\sharp}_{\Sigma} \to \Gamma\backslash D_\Sigma$ and $\Gamma\backslash D^{\sharp}_{\Sigma,\mathrm{val}} \to \Gamma\backslash D_{\Sigma,\mathrm{val}}$ are proper surjective maps, described by logarithmic structures, whose fibers are products of finite copies of $\mathbf{S}^1$.

In the classical case $D = \mathfrak{h}_g$, we have $D_{\mathrm{SL}(2)} = D_{\mathrm{BS}}$ and $D_{\mathrm{SL}(2),\mathrm{val}} = D_{\mathrm{BS},\mathrm{val}}$ in Fundamental Diagram. This is because, for such simple polarizes Hodge structures, all filters appeared in the associated set of monodoromy-weight filtrations are totally linearly ordered by inclusion and the Griffiths transversality becomes a vacant condition. Fundamental Diagram gives a relation between toroidal compactifications $\Gamma\backslash D_\Sigma$ of $\Gamma\backslash\mathfrak{h}_g$ and the Borel-Serre compactification $\Gamma\backslash D_{\mathrm{BS}}$ of $\Gamma\backslash\mathfrak{h}_g$. The Satake-Baily-Borel compactification sits under $\Gamma\backslash D_{\mathrm{SL}(2)} = \Gamma\backslash D_{\mathrm{BS}}$. Already in this classical case, these relations were not known before.

In this book, we study all these eight spaces. To prove the main theorem in Subject I and to prove that $\Gamma\backslash D_\Sigma$ has good properties such as Hausdorff property, nice infinitesimal calculus, etc., we need to consider all the eight spaces; we discuss from the right to the left in the Fundamental Diagram to deduce nice properties of $\Gamma\backslash D_\Sigma$, starting from nice properties of the Borel-Serre compactifications in [BS].

Added this time. [U2], [U3] are some geometric applications of the present results. A new project [KNU], which is an evolution for logarithmic *mixed* Hodge structures, is in progress.

Acknowledgment. The speaker would like to express his gratitude to the organizers of 8th Oka Symposium.

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§1. Log Hodge Theory (with K. Kato)

Degenerating elliptic curve

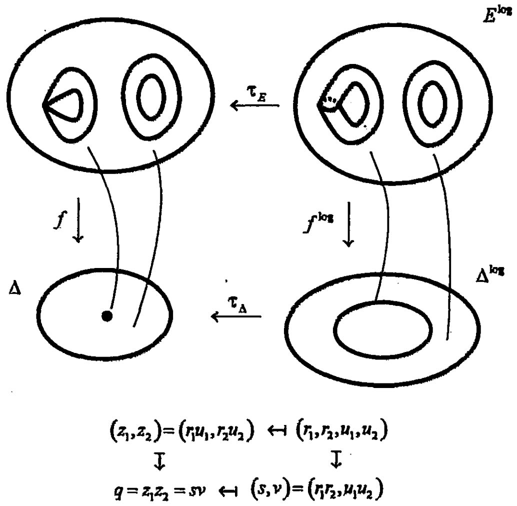

Figure 1

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$f : E \to \Delta,$

$q \in \Delta^*,\ f^{-1}(q) = \mathbb{C}^\times/q^{\mathbb{Z}} \cong \underset{\substack{\| \\ X}}{\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z})}\ (\tau \in \mathfrak{h}),$

$f^{-1}(0) = \mathbb{P}^1(\mathbb{C})/(0\sim\infty).$

$H_1(X,\mathbb{Z}) = \mathbb{Z}\tau + \mathbb{Z}, \quad \tau, 1 \leftrightarrow \gamma_1, \gamma_2 \quad \mathbb{Z}$-basis,

$H^1(X,\mathbb{Z}), \quad e_1, e_2$ dual basis

z : coord. of $\mathbb{C}$.

(cohomology class of dz) $= \tau e_1 + e_2$

$\because \int_{\gamma_1} dz = \tau, \quad \int_{\gamma_2} dz = 1.$

$f' : E^* \to \Delta^*$, the restriction.

$H'_{\mathbb{Z}} := R^1 f'_* \mathbb{Z}, \quad \mathcal{M}' := R^1 f'_*(\Omega^\bullet_{E^*/\Delta^*}).$

Then, $\boxed{\mathcal{M}' = \mathcal{O}_{\Delta^*} \otimes_{\mathbb{Z}} H'_{\mathbb{Z}}.}$

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$R^1 f_* \mathbb{Z}$ is not locally constant on Δ.

$(R^1 f_* \mathbb{Z})_0 = H^1(f^{-1}(0), \mathbb{Z})$ is of rank 1.

e_1 extends over $0 \in \Delta$, but $\overset{\text{vanishing cycle}}{e_2}$ does not.

$\mathcal{M} := R^1 f_*(\omega^\bullet_{E/\Delta}), \quad \omega^\bullet_{E/\Delta} = \Omega^\bullet_{E/\Delta}(\log f^{-1}(0)).$

$= \mathcal{O}_\Delta e_1 \oplus \mathcal{O}_\Delta \omega, \quad \left(\omega := \frac{\log q}{2\pi i} e_1 + e_2\right)$

$$\begin{array}{ccc} X & \leftarrow & X^{\log} \\ \downarrow f & & \downarrow f^{\log} \\ \Delta & \xleftarrow{\tau} & \Delta^{\log}, \end{array} \quad q \leftarrow\!\shortmid \left(|q|, \frac{q}{|q|}\right)$$

$H_{\mathbb{Z}} := R^1 f^{\log}_* \mathbb{Z}$ is locally constant on $\Delta^{\log}$.

$j^{\log} : \Delta^* \hookrightarrow \Delta^{\log},$

$j^{\log}_*(\mathcal{O}_{\Delta^*}) \supset \mathcal{O}^{\log}_\Delta := \tau^{-1}(\mathcal{O}_\Delta)[\log q].$

Then, $\boxed{\mathcal{O}^{\log}_\Delta \otimes_{\tau^{-1}(\mathcal{O}_\Delta)} \tau^{-1}(\mathcal{M}) = \mathcal{O}^{\log}_\Delta \otimes_{\mathbb{Z}} H_{\mathbb{Z}}.}$

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Log structure

$(X, \mathcal{O}_X)$: local ringed space

Log structure is $\alpha : M_X \to \mathcal{O}_X$ homomorphism of sheaves of monoids s.t. $\alpha^{-1}(\mathcal{O}_X^\times) \cong \mathcal{O}_X^\times$.

Example $X = \Delta^n \supset D = \{q_1 \cdots q_n = 0\}$.

$M_X = \{f \in \mathcal{O}_X \mid f \text{ is invertible on } \Delta^n - D\} \overset{\alpha}{\hookrightarrow} \mathcal{O}_X$

$= \coprod_{e_j \in \mathbb{N}} \mathcal{O}_X^\times q_1^{e_1} \cdots q_n^{e_n}$

$= \mathcal{O}_X^\times \times \mathbb{N}^n$

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$(X^{\log}, \mathcal{O}_X^{\log})$

monoid $\mathcal{S}$ is fs if
- finitely generated.
- integral: $ab = ac \Rightarrow b = c$. (Then $\mathcal{S} \subset \mathcal{S}^{gp}$).
- saturated: $a \in \mathcal{S}^{gp}$, $a^n \in \mathcal{S}$ $(\exists n \geqslant 1) \Rightarrow a \in \mathcal{S}$.

X : local ringed space,
$\mathcal{S}$: fs monoid, constant sheaf on X.
$\alpha_1 : \mathcal{S} \to \mathcal{O}_X$ homomorphism.

$$\begin{tikzcd}
\alpha_1^{-1}(\mathcal{O}_X^*) \arrow[r, hook] \arrow[d, "\alpha_1"'] \arrow[dr, phantom, "\square"] & \mathcal{S} \arrow[d, dashed] \arrow[dd, bend left, "\alpha_1"] \\
\mathcal{O}_X^* \arrow[r, dashed] \arrow[dr, hook] & \tilde{\mathcal{S}} \arrow[d, dashed, "\alpha"] \\
 & \mathcal{O}_X
\end{tikzcd}$$

pushout

associated log structure

A log structure on X is fs if it is locally isomorphic to the above.

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toric variety

Example. $\mathcal{S}$: fs monoid
$X = \mathrm{Spec}(\mathbb{C}[\mathcal{S}])_{an} = \mathrm{Hom}(\mathcal{S}, \mathbb{C}^{mult})$
$\alpha_1 : \mathcal{S} \to \mathbb{C}[\mathcal{S}] \to \mathcal{O}_X$.
Then $\underbrace{\tilde{\mathcal{S}}}_{M_X :=}$ is an fs log structure on X.

Convenient definition of $(X^{\log}, \mathcal{O}_X^{\log})$.
toric case: $X = \mathrm{Spec}(\mathbb{C}[\mathcal{S}])_{an}$.
$X^{\log} := \mathrm{Hom}(\mathcal{S}, \mathbb{R}_{\geq 0}^{mult} \times S^1) \xrightarrow{\tau} X$,
$j^{\log} : U = \mathrm{Spec}(\mathbb{C}[\mathcal{S}^{gp}])_{an} \hookrightarrow X^{\log}$,
$\mathcal{O}_X^{\log} \subset j^{\log}_*(\mathcal{O}_U)$: sheaf of $\tau^{-1}(\mathcal{O}_X)$-subalgebras generated by log of local sections of M_X^{gp}.

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general case: X : fs log local ringed space.
Locally on X, given $\mathcal{S} \to \mathcal{O}_X$ s.t. $M_X = \tilde{\mathcal{S}}$.

$$\begin{array}{l} X^{\log} = X \times_{\mathrm{Hom}(\mathcal{S}, \mathbb{C}^{mult})} \mathrm{Hom}(\mathcal{S}, \mathbb{R}_{\geq 0}^{mult} \times S^1) \\ \downarrow \tau \\ X \end{array}$$

(fiber product as topological space),

$\mathcal{O}_X^{\log} = \mathcal{O}_X \otimes_{\mathcal{O}_Z} \mathcal{O}_Z^{\log}$, where $Z = \mathrm{Spec}(\mathbb{C}[\mathcal{S}])_{an}$.

Note. $f_j \in M_{X,x}^{gp}$ s.t. $(f_j \bmod \mathcal{O}_{X,x}^*)_{1 \leq j \leq n}$ is a basis of $(M_X^{gp}/\mathcal{O}_X^*)_x$.

Then for $y \in \tau^{-1}(x)$,
$\mathcal{O}_{X,x}[T_1, \dots, T_n] \xrightarrow{\sim} \mathcal{O}_{X,y}^{\log}$, $T_j \mapsto \log f_j$.
Not a local ring!

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Define
$sp(y) = \{\underbrace{s : \mathcal{O}_{X,y}^{\log} \to \mathbb{C},\ \mathbb{C}\text{-alg. homom.}}_{s(f) = f(x)\ (f \in \mathcal{O}_{X,x})}\}$.
For a fixed $s_0 \in sp(y)$,
$sp(y) \xrightarrow{\sim} \mathrm{Hom}((M_X^{gp}/\mathcal{O}_X^*)_x, \mathbb{C}^{add})$.
$s \mapsto (M_{X,x}^{gp} \ni f \mapsto s(\log f) - s_0(\log f) \in \mathbb{C})$

For an $\mathcal{O}_X^{\log}$-module $\mathcal{P}$, $y \in X^{\log}$, $s \in sp(y)$,
define $\mathcal{P}(s) := \mathbb{C} \otimes_{s, \mathcal{O}_{X,y}^{\log}} \mathcal{P}_y$. $\mathbb{C}$-module.

<u>Example</u> (log point of log rank 1)

$x = \mathrm{Spec}\, \mathbb{C}$ with

$$M_x = \mathbb{C}^\times \times \mathbb{N} = \coprod_{m \in \mathbb{N}} \mathbb{C}^\times q^m \xrightarrow{\alpha} \mathbb{C}.$$
$$c q^m \mapsto c\, 0^m$$

$x^{\log} = S^1$, $\mathcal{O}_x^{\log} = \mathbb{C}[l]$ $(l = \log q)$,

$\omega_x^{1,\log} = \mathbb{C}[l]\, dl$,

$y \in x^{\log}$, $\mathrm{sp}(y) = \mathbb{C}$

$\check{\varphi} : \check{E}_\sigma \to \Gamma(\sigma)^{gp} \backslash \check{D}_{\mathrm{orb}}$

$\Gamma < G_{\mathbb{Z}} = \mathrm{Aut}(H_0, \langle\, ,\, \rangle_0)$ subgroup.

$\mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}\, G_{\mathbb{R}}$.

$$\check{D}_{\mathrm{orb}} := \left\{ (\sigma, Z) \;\middle|\; \begin{array}{l} \sigma : \text{nilpotent cone in } \mathfrak{g}_{\mathbb{R}} \\ Z : \exp(\sigma_{\mathbb{C}})\text{-orbit in } \check{D} \end{array} \right\}.$$

(We restrict this by imposing conditions such as rationality of σ, σ's forming a fan, Griffiths transversality, positivity.)

σ : rational nilpotent cone in $\mathfrak{g}_{\mathbb{R}}$, i.e.
$\sigma = \mathbb{R}_{\geq 0} \log(\Gamma(\sigma))$, where $\Gamma(\sigma) := \Gamma \cap \exp(\sigma)$.

We define <u>universal set-theoretic period map</u>

$$\check{\varphi} : \check{E}_\sigma := \mathrm{toric}_\sigma \times \check{D} \to \Gamma(\sigma)^{gp} \backslash \check{D}_{\mathrm{orb}},$$

$$\check{\varphi}(q, F) := ((\sigma(q), \exp(\sigma(q)_{\mathbb{C}}) \exp(z) F) \bmod \Gamma(\sigma)^{gp}),$$

where

$\mathrm{toric}_\sigma := \mathrm{Spec}(\mathbb{C}[\Gamma(\sigma)^\vee])_{an} = \mathrm{Hom}(\Gamma(\sigma)^\vee, \mathbb{C}^{\mathrm{mult}})$,

$\mathrm{torus}_\sigma := \mathrm{Spec}(\mathbb{C}[\Gamma(\sigma)^{\vee gp}])_{an} = \mathrm{Hom}(\Gamma(\sigma)^{\vee gp}, \mathbb{C}^\times)$
$\qquad = \mathbb{C}^\times \otimes \Gamma(\sigma)^{gp}$.

$\pi_1(\mathrm{toric}_\sigma^{\log}) \simeq \pi_1(\mathrm{torus}_\sigma) \simeq \Gamma(\sigma)^{gp}$.

For $q \in \mathrm{toric}_\sigma$ endowed with log structure,

$\pi_1^+(q^{\log}) := \pi_1(q^{\log}) \cap \Gamma(\sigma)$,

$\sigma(q) := \mathbb{R}_{\geq 0} \log(\pi_1^+(q^{\log}))$.

$\mathfrak{F} := \Gamma(\sigma)^\vee$, $\mathfrak{F}' := \{ f \in \mathfrak{F} \mid f(q) \neq 0 \}$.

$$\begin{array}{ccc} z \in \sigma_{\mathbb{C}} & \to & \sigma_{\mathbb{C}} / \log(\Gamma(\sigma)^{gp}) \\ \| & & \downarrow\wr \\ \sum z_j N_j & & \mathrm{torus}_\sigma = \mathrm{Hom}(\mathfrak{F}^{gp}, \mathbb{C}^\times) \longrightarrow \mathrm{Hom}((\mathfrak{F}')^{gp}, \mathbb{C}^\times). \\ \downarrow & & \qquad\qquad \Psi \qquad\qquad\qquad\qquad\qquad \Psi \\ & & \prod \exp(2\pi i z_j) \otimes \exp(N_j) \longmapsto (\text{evaluation at } q) \end{array}$$

<u>$\Gamma \backslash D_\Sigma$</u>

$\sigma = \sum_{j=1}^{n} \mathbb{R}_{\geq 0} N_j$: a rational nilpotent cone in $\mathfrak{g}_{\mathbb{R}}$.

$Z = \exp(\sigma_{\mathbb{C}}) F \subset \check{D}$ is a σ-<u>nilpotent orbit</u>

if (1) $N F^p \subset F^{p-1}$ $(\forall p, \forall N \in \sigma)$.

(2) $\exp(\sum z_j N_j) F \in D$ if $\mathrm{Im}(z_j) \gg 0$.

Define

$D_\sigma := \{ (\tau, Z) \mid \tau \prec \sigma,\ Z \subset \check{D} : \tau\text{-nilpotent orbit} \}$.

$E_\sigma := \check{\varphi}^{-1}(D_\sigma) \subset \check{E}_\sigma$ (slits appear)

$\varphi := \check{\varphi}|_{E_\sigma} : E_\sigma \to \Gamma(\sigma)^{gp} \backslash D_\sigma$.

$\mathrm{toric}_\sigma \times \check{D}$ (e.g. I, p.7)

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Category $\mathcal{B}$

Z : analytic space, $S \subset Z$: subset,

Define

$U \subset S$ is open in the <u>strong topology</u>

$\Leftrightarrow$ For $\forall Y$: analytic space, and

$\forall \lambda : Y \to Z$ morphism s.t. $\lambda(Y) \subset S$,

$\lambda^{-1}(U)$ is open in Y.

$\mathcal{B}$: category of local ringed spaces X over $\mathbb{C}$ s.t.

$\exists (U_\lambda)_\lambda$: open covering of X,

$U_\lambda \simeq {}^\exists S_\lambda \overset{\text{subset}}{\subset} {}^\exists Z_\lambda$: analytic space,

where S_λ is endowed with the strong topology

and the inverse image of $\mathcal{O}_{Z_\lambda}$.

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$\mathcal{A}$: category of analytic spaces.

Define $\mathcal{A}(\log)$ (resp. $\mathcal{B}(\log)$) as the category of objects of $\mathcal{A}$ (resp. $\mathcal{B}$) endowed with an fs log structure.

A <u>log manifold</u> is a "smooth object" in $\mathcal{B}(\log)$.

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Theorem A

$$\left.\begin{array}{l}\Gamma : \text{a neat subgroup of } G_{\mathbb{Z}} \\ \Sigma : \text{a fan in } \mathfrak{g}_{\mathbb{Q}}\end{array}\right\} \underline{\text{strongly compatible}}$$

$$\Leftrightarrow \begin{cases} \gamma \in \Gamma,\ \sigma \in \Sigma \Rightarrow \mathrm{Ad}(\gamma)\sigma \in \Sigma, \\ \sigma \in \Sigma \Rightarrow \sigma = \mathbb{R}_{\geq 0} \log(\Gamma(\sigma)). \end{cases}$$

$$\begin{array}{ll} E_\sigma \subset \check{E}_\sigma = \text{toric}_\sigma \times \check{D} & (\sigma \in \Sigma) \\ \varphi \downarrow \ \sigma_{\mathbb{C}}\text{-torsor} & \text{w.r.t. the strong} \\ \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma & \text{topology on } E_\sigma \\ \quad \downarrow \ \text{locally isomorphic} & \text{and its quotient} \\ \Gamma \backslash D_\Sigma & \text{topologies.} \end{array}$$

$\mathcal{O}$, M are induced along this diagram.

<u>Theorem A</u>. $\Gamma \backslash D_\Sigma$ is a Hausdorff log manifold.

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PLH

$X \in \mathcal{B}(\log)$,

$H = (H_{\mathbb{Z}}, \langle\ ,\ \rangle, F)$: a pre-PLH on X,

H is a <u>polarized log Hodge structure</u> if, $\forall x \in X$

(1) $(d \otimes 1_{H_{\mathbb{Z}}})(F^p|_{X^{\log}}) \subset \omega_X^{1,\log} \otimes_{\mathcal{O}_X^{\log}} (F^{p-1}|_{X^{\log}})$ $(\forall p)$

(2) Let $f_j \in M_{X,x}$ $(1 \leq j \leq n)$ generates $(M_X/\mathcal{O}_X^\times)_x$

$y \in X^{\log}$, $s \in sp(y)$.

Then $(H_{\mathbb{Z},y}, \langle\ ,\ \rangle_y, F(s))$ is a PH

if $\exp(s(\log f_j))$ is sufficiently near 0.

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Key

<u>Proposition</u>. x : an fs log point,

$H = (H_{\mathbb{Z}}, \langle\ ,\ \rangle, F)$: a pre-PLH on x.

$y \in x^{\log}$,

$h_j \in \mathrm{Hom}(M_x^{gp}/\mathcal{O}_x^\times, \mathbb{Z}) \quad (1 \leq j \leq n)$,

$\updownarrow \qquad\qquad \|$

$\gamma_j \in \pi_1(x^{\log})$.

$N_j = \log(\gamma_j) : H_{\mathbb{Q},y} \to H_{\mathbb{Q},y}$.

$z_j \in \mathbb{C}$,

$s_0,\ s \in sp(y)$ s.t. for $^\forall f \in M_x^{gp}$,

$$s\left(\frac{\log f}{2\pi i}\right) - s_0\left(\frac{\log f}{2\pi i}\right) = \sum_j z_j h_j(f).$$

Then

$$\boxed{\begin{array}{c} F(s) = \exp(\Sigma z_j N_j) F(s_0), \\ \text{(a PLH on an fs log point)} = \text{(a nilpotent orbit)} \end{array}}$$

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<u>Moduli functor</u>

$\Phi = (w, (h^{p,q})_{p,q}, H_0, \langle\ ,\ \rangle_0, \Gamma, \Sigma)$

$X \in \mathcal{B}(\log)$

$\mu \in H^0(X, \Gamma \backslash \mathcal{I}som((H_{\mathbb{Z}}, \langle\ ,\ \rangle), \underbrace{(H_0, \langle\ ,\ \rangle_0)}_{\text{considered as a constant sheaf}}))$

$\uparrow$

<u>Γ-level structure</u>

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$\underline{PLH}_\Phi(X) := \left\{ \begin{array}{l} \text{PLH } H \text{ on } X \text{ of type } (w, (h^{p,q})) \text{ with a} \\ \Gamma\text{-level str. } \mu \text{ satisfying (1), (2) below} \end{array} \right\}$,

$\forall x \in X$, $^\forall y \in x^{\log}$,

$\tilde{\mu}_y : (H_{\mathbb{Z},y}, \langle\ ,\ \rangle_y) \xrightarrow{\sim} (H_0, \langle\ ,\ \rangle_0)$ a lifting of μ_y

(1) $^\exists \sigma \in \Sigma$ s.t.

$\mathrm{Im}(\pi_1^+(x^{\log}) \to \mathrm{Aut}(H_{\mathbb{Z},y}) \xrightarrow{\text{by } \tilde{\mu}_y} \mathrm{Aut}(H_0)) \subset \exp(\sigma)$

(2) For $\sigma \in \Sigma$: smallest such, $s \in sp(y)$,

$\exp(\sigma_{\mathbb{C}}) \tilde{\mu}_y(F(s))$ is a σ-nilpotent orbit

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<u>Moduli</u>

<u>Theorem B</u>. (i) $\exists \varphi : \underline{PLH}_\Phi \xrightarrow{\sim} \mathrm{Mor}(\ , \Gamma\backslash D_\Sigma)$.

(ii) For $^\forall Z$: a local ringed space over $\mathbb{C}$ with a log structure,

$$\begin{tikzcd}
\underline{PLH}_\Phi|_{\mathcal{A}(\log)} \arrow[r, "\varphi", "\sim"'] \arrow[dr, "\forall h"'] & \mathrm{Mor}(\ , \Gamma\backslash D_\Sigma)|_{\mathcal{A}(\log)} \arrow[d, dashed, "f"] & \Gamma\backslash D_\Sigma \arrow[d, "\exists! f"] \arrow[l, phantom, "\longleftarrow"] \\
& \mathrm{Mor}(\ , Z)|_{\mathcal{A}(\log)} & Z.
\end{tikzcd}$$

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Extension of period map [Kato-Nakayama-U]

$X \overset{\text{locally}}{\simeq}$ open $\subset$ toric, $U = X_{\text{triv}}$, $u \in U$.

H : VPH on U, unipotent.

$G_{\mathbb{Z}} \supset \Gamma$: neat $\supset \operatorname{Im}(\pi_1(U,u) \to G_{\mathbb{Z}})$.

$\varphi : U \to \Gamma\backslash D$.

Theorem (i) (codim 1) Case $X - U$ smooth divisor,

$$\begin{tikzcd} U \arrow[d, "\varphi"'] & \subset & X \arrow[d] \\ \Gamma\backslash D & \subset & \Gamma\backslash D_{\Sigma}. \end{tikzcd}$$

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(ii) (local) $\forall x \in X$, $\exists W \subset X$: nbd of x,

$$\begin{tikzcd} U \arrow[d, "\varphi"'] & \supset & U \cap W \arrow[d] & \subset & W' \arrow[d] \\ \Gamma\backslash D & \leftarrow & \Gamma'\backslash D & \subset & \Gamma'\backslash D_{\Sigma}. \end{tikzcd}$$

W' ← log modification of W

(iii) (global) Assume $\exists \Sigma$: complete fan.

$$\begin{tikzcd} U \arrow[d, "\varphi"'] & \subset & X(\Sigma) \arrow[d] \\ \Gamma\backslash D & \subset & \Gamma\backslash D_{\Sigma} \end{tikzcd}$$

$X(\Sigma)$ ← log modification associated to Σ

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Σ : complete fan (C. Nakayama)

$\overset{\text{def}}{\Leftrightarrow} \bigcup_{\sigma \in \Sigma} \sigma = \bigcup_{\sigma \in N} \sigma.$

$N := \left\{ \sigma \;\middle|\; \begin{array}{l} \sigma : \text{rational nilpotent cone in } \mathfrak{g}_{\mathbb{R}}, \\ (\sigma, Z) : \text{a nilpotent orbit for some } Z \subset \check{D} \end{array} \right.$

Note For old definition,

K. Watanabe, A counterexample to a conjecture of complete fan, J. Math. Kyto Univ. 48-4 (200

Image of extended period map

M_H^{an} : compactified moduli of algebraic varieties of general type

$\cup$

M : an irreducible compont.

Theorem Assume $\exists \Sigma$: a complete fan.

Take $M' \to M$ finite covering + blowing-ups.

Then $\exists \varphi : M' \to \Gamma\backslash D_{\Sigma}$ log period map.

$\varphi(M') \subset \Gamma\backslash D_{\Sigma}$ analytic subspace, Moishezon.

S. Usui, J. Alg. Geom. 15 (2006)

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Sketch of Proof

weakly semistable reduct. [AK97e]

$$\begin{tikzcd} X_0 \arrow[d] & X \arrow[l] \arrow[d, "f"'] \arrow[r, dashed] & \mathrm{Proj}_S R_{X/S} \arrow[dl] \\ \mathrm{Hilb}_H \supset S_0 & S \arrow[l] & \end{tikzcd}$$

this yields boundedness of M_H^{an}

[Karu 00] + [BCHM^{c}K 06e], [Siu 06e]

Birkar-Cascini-Hacon-McKernan

By [K. Kato 89], $f : X \to S$ log smooth morphism of log smooth fs log analytic spaces + ...

Apply [KMN 02]: $t_0 \in S^{\log}$, $(H_0, \langle\, ,\rangle_0) := (H_{\mathbb{Z}, t_0}, \langle\, ,\rangle_{t_0})$

$$\begin{cases} H_{\mathbb{Z}} = R^w f_*^{\log} \mathbb{Z} / \text{torsion} \\ \mathcal{M} = R^w f_*(\omega^{\bullet}_{X/S}), \quad \mathcal{M}^p = R^w f_*(\omega^{\geq p}_{X/S}) \\ \Gamma = \operatorname{Im}(\pi_1(S^{\log}) \to \operatorname{Aut}(H_0, \langle\, ,\rangle_0)) \\ \Sigma : \text{a fan in } \mathfrak{g}_{\mathbb{R}} \text{ below (Assume } \exists) \end{cases}$$

form a PLH on S of type Φ

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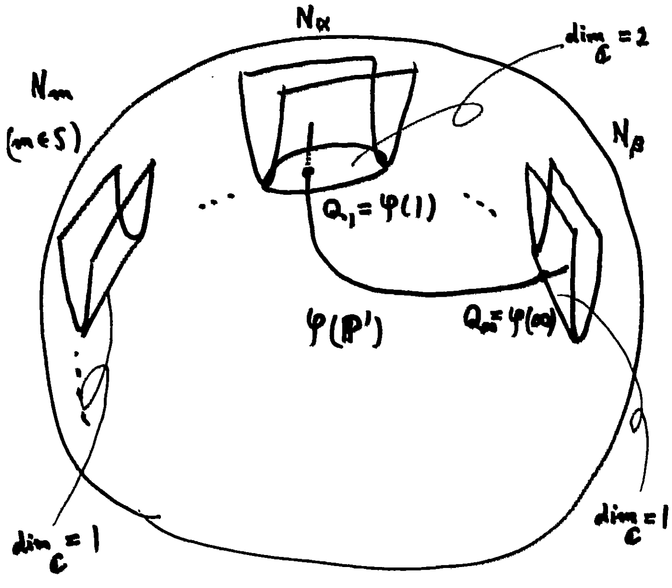
§2. Quintic-mirror



$\Gamma \backslash D_{\Xi}$, $\dim_{\mathbb{C}} = 4$

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Quintic-mirror family.

- D. Morrison 93, J.AMS,
- Candelas - de la Ossa - Green - Parkes 91, Phys. Lett.

$$Q_\psi = \left(\sum_{j=1}^{5} x_j^5 - 5\psi x_1 \cdots x_5 = 0\right) \subset \mathbb{P}^4 \quad (\psi \in \mathbb{P}^1)$$

Sing Q_ψ:

$\psi^5 = 1$, $(\alpha_1, \dots, \alpha_5)$, $\alpha_j^5 = 1$, $\psi \alpha_1 \cdots \alpha_5 = 1$

($5^3 = 125$ ordinary double points.)

$\psi = \infty$, $Q_\infty = \bigcup (x_j = 0)$

(4-dimensional simplex.)

$\mu_5 := \{\alpha \in \mathbb{C} \mid \alpha^5 = 1\}$.

$G = \{\alpha \in (\mu_5)^5 \mid \alpha_1 \cdots \alpha_5 = 1\} / (\text{diagonal})$.

coordinate-wise action on $\mathbb{P}^4$.

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Q_ψ / G canonical singularities appear: cA_4 under each $(x_j = x_k = 0)$, three of them intersect under each point in $(x_j = x_k = x_l = 0)$, hol. 3-form on Q_ψ is G-invariant by adjunction.

$\uparrow$

W_ψ minimal resolution of these quotient sing simultaneously for every $\psi \in \mathbb{C}$. hol. 3-form extends to a nowhere vanishing form on W_ψ.

W_ψ : singular $\Leftrightarrow$ $\psi^5 = 1$, $\psi = \infty$.

For smooth W_ψ, $h^{p,q} = 1$ $(p+q = 3)$.

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by $\alpha \in \mu_5$, $(x_1, \dots, x_5) \mapsto (\alpha^{-1} x_1, x_2, \dots, x_5)$,

$W_{\alpha\psi} \cong W_\psi$.

Let $\lambda = \psi^5$, and

$$\begin{array}{ccc} (W_\psi)/\mu_5 & = & (W_\lambda) \\ \downarrow & & \downarrow \\ (\psi\text{-plane})/\mu_5 & = & (\lambda\text{-plane}) \end{array}$$

Quintic-mirror family.

$\Gamma := \mathrm{Im}\left(\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}) \to G_{\mathbb{Z}}\right)$

λ-plane

fix a base point $b \in \mathbb{P}^1 - \{0, 1, \infty\}$ and let $H^3(W_b, \mathbb{Z}) = H_0$.

Equation of quintic mirror family

$$\Big(\sum_{j=1}^{5} y_j\Big)^5 - 5^5 \lambda\, y_1 \cdots y_5 = 0,$$

$$y_j := x_j^5, \qquad \lambda := \psi^5.$$

PLH [KU]

In the present case,

$$w = 3, \quad h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1.$$

$$H_0 = \bigoplus_{j=1}^{4} \mathbb{Z} e_j, \quad \langle e_3, e_1\rangle_0 = \langle e_4, e_2\rangle_0 = 1.$$

Types of monodromy logarithms

$$0: \quad \underset{(3,0)}{\bullet} \quad \underset{(2,1)}{\bullet} \quad \underset{(1,2)}{\bullet} \quad \underset{(0,3)}{\bullet} \qquad e_j \mapsto 0 \quad (1 \le j \le 4)$$

$$N_\alpha: \quad \underset{(3,0)}{\bullet} \quad \begin{array}{c} \underset{(2,2)}{\bullet} \\ \downarrow \\ \underset{(1,1)}{\bullet} \end{array} \quad \underset{(0,3)}{\bullet} \qquad e_3 \mapsto e_1 \mapsto 0 \qquad \underbrace{e_2,\ e_4}_{\mapsto\, 0}$$

$$N_\beta: \quad \begin{array}{c} \underset{(3,3)}{\bullet} \\ \downarrow \\ \underset{(2,2)}{\bullet} \\ \downarrow \\ \underset{(1,1)}{\bullet} \\ \downarrow \\ \underset{(0,0)}{\bullet} \end{array} \qquad e_4 \mapsto e_3 \mapsto -e_1 \mapsto e_2 \mapsto 0$$

$$N_m: \quad \begin{array}{c} \underset{(3,1)}{\bullet} \\ \downarrow \\ \underset{(2,0)}{\bullet} \end{array} \quad \begin{array}{c} \underset{(1,3)}{\bullet} \\ \downarrow \\ \underset{(0,2)}{\bullet} \end{array} \qquad e_1 \mapsto e_3 \qquad e_4 \mapsto -m e_2$$

$m \in S :=$ (square free positive integers).

Let

$$\sigma_\mu = \mathbb{R}_{\ge 0} N_\mu \quad \text{for } \mu = \alpha, \beta, m.$$

Prop. [KU] $\Xi :=$ (rat. nilp. cones of rk 1 in $\mathfrak{g}_{\mathbb{R}}$)

$= \{ \mathrm{Ad}(g)\sigma \mid \sigma = \{0\}, \sigma_\alpha, \sigma_\beta, \sigma_m\ (m \in S),\ g \in G_{\mathbb{Q}} \}$.

This is a complete fan, i.e.,

$(\sigma, {}^{\exists}Z)$: nilp. orbit $\Rightarrow \sigma \in \Xi$.

$D_\Xi = \{ (\sigma, Z) \text{ nilp. orbit} \mid \sigma \in \Xi,\ Z \subset \check{D} \}$.

Γ : a (neat) subgroup of $G_{\mathbb{Z}}$ of finite index,

For $\sigma \in \Xi$, $\quad \Gamma(\sigma) = \Gamma \cap \exp(\sigma)$.

If $\sigma = 0$, $\quad D \simeq \{ (0, F) \mid F \in D \} \subset D_\Xi$.

If $\sigma \ne 0$, $\quad \Gamma(\sigma) \simeq \mathbb{N}$. γ : generator, $N = \log \gamma$

Def.

(σ, Z) is a **nilpotent orbit** if

$Z = \exp(\mathbb{C}N)F$, $NF^p \subset F^{p-1}$ ($\forall p$) and

$\exp(iyN)F \in D$ ($y \gg 0$) hold

for $\mathbb{R}_{\ge 0} N = \sigma$ and $F \in Z$.

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Log manifold $\Gamma\backslash D_\Xi$

$$E_\sigma = \left\{ (q, F) \in \mathbb{C}\times\check{D} \;\middle|\; \begin{array}{l} \exp\left(\frac{\log q}{2\pi i}N\right)F \in D \text{ if } q \neq 0, \\ \exp(\mathbb{C}N)F \text{ is } \sigma\text{-nilp. orbit if } q = 0 \end{array} \right\}$$

(1) $E_\sigma \to \Gamma(\sigma)^{gp}\backslash D_\sigma$,

$$(q,F) \mapsto \begin{cases} \exp\left(\frac{\log q}{2\pi i}N\right)F & \text{if } q \neq 0, \\ (\sigma, \exp(\mathbb{C}N)F) & \text{if } q = 0. \end{cases}$$

$\mathbb{C}\times\check{D} \supset \{0\}\times\check{D}$.

associated log structure

$$M = \coprod_{n\geq 0} \mathcal{O}^\times q^n \cong \mathcal{O}^\times \times \mathbb{N}.$$

$\downarrow$

$\mathcal{O}$

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Strong topology of E_σ in $\mathbb{C}\times\check{D}$

$U \subset E_\sigma$ is open

$\Leftrightarrow \begin{cases} \forall f : \forall Y \to \mathbb{C}\times\check{D} \text{ mor. of anal. sps s.t. } f(Y) \subset E_\sigma, \\ f^{-1}(U) \text{ is open in } Y. \end{cases}$

Via (1), introduce on $\Gamma(\sigma)^{gp}\backslash D_\sigma$ topology, $\mathcal{O}$, M.

Introduce these structures on $\Gamma\backslash D_\Xi$ so that $\Gamma(\sigma)^{gp}\backslash D_\sigma \to \Gamma\backslash D_\Xi$ ($\sigma\in\Xi$) form an open cover.

Then, $\Gamma\backslash D_\Xi$ is a "log manifold"

≒ log analytic space, but ∃ "slits".

In fact, $\dim_{\mathbb{C}} D = 4$.

$$\dim_{\mathbb{C}}(\Gamma(\sigma)^{gp}\backslash(D_\sigma - D)) = \begin{cases} 2 & \text{if } \sigma = \sigma_\alpha, \\ 1 & \text{if } \sigma = \sigma_\beta, \\ 1 & \text{if } \sigma = \sigma_m. \end{cases}$$

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Period map

$$\begin{tikzcd} z \arrow[d, mapsto] & \in & \mathfrak{h} \arrow[r, "\tilde{\varphi}"] \arrow[d] & D \arrow[d] \\ q = e^{2\pi i z} & \in & \Delta^* \arrow[r, "\varphi"] \arrow[ur, "\psi"] & \langle\gamma\rangle\backslash D \end{tikzcd}$$

$N := \log\gamma$

$\exp(-zN)\,\tilde{\varphi}(z) =: \psi(q)$

Nilpotent orbit theorem of Schmid:

$\exists\, \psi(0) =: F$, $\quad \tilde{\varphi}(z) \sim \exp(zN)F \quad (\mathrm{Im}\, z \to \infty)$.

In our space $\langle\gamma\rangle\backslash D_\sigma$ $\quad (\sigma := \mathbb{R}_{\geq 0}N)$:

$\exp(zN)F \to (\sigma, \exp(\sigma_{\mathbb{C}})F) \bmod \gamma \quad (\mathrm{Im}\, z \to \infty)$.

Hence, for $\exists\, g(z) \in G_{\mathbb{R}}$,

$\varphi(q) = g(z)\exp(zN)F \to (\sigma, \exp(\sigma_{\mathbb{C}})F) \bmod \gamma$.

$(q \to 0)$

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Period map extends to

$\varphi : \mathbb{P}^1 \to \Gamma\backslash D_\Xi$

$0 \mapsto$ point $\in \Gamma\backslash D$

$1 \mapsto \mathrm{Ad}(g)\,\sigma_\alpha$-nilp. orbit $\quad (\exists\, g \in G_{\mathbb{Q}})$

$\infty \mapsto \mathrm{Ad}(g)\,\sigma_\beta$-nilp. orbit $\quad (\exists\, g \in G_{\mathbb{Q}})$

Generic

Global Torelli Theorem.

φ is the normalization over its image.

(even as log analytic spaces.)

Outline of Proof

① At $P_1 := 1 \in \mathbb{P}^1$, $Q_1 := \varphi(P_1) \in \Gamma\backslash D_\Sigma$. $\varphi^{-1}(Q_1) = \{P_1\}$.

$N := N_\alpha$, $N(e_3) = e_1$, $N(e_j) = 0$ $(j \neq 3)$.

$\exp(-zN)e_3 = e_3 - z e_1$ simple-valued.

Let $F^3(\tilde{q}) = \mathbb{C}\,\omega(\tilde{q})$, $\omega(\tilde{q}) = \sum_{j=1}^{4} a_j(\tilde{q})\, e_j$.

$$t := \frac{a_1(\tilde{q})}{-a_3(\tilde{q})} = \frac{\langle e_3, \omega(\tilde{q})\rangle_0}{\langle e_1, \omega(\tilde{q})\rangle_0}$$

$$= \frac{\langle \exp(-zN)e_3, \omega(\tilde{q})\rangle_0 + z\langle e_1, \omega(\tilde{q})\rangle_0}{\langle e_1, \omega(\tilde{q})\rangle_0}$$

$= z +$ (simple-valued fun. in $\tilde{q}$).

$\therefore\ q := e^{2\pi i t} = u\tilde{q}$ $\quad(\tilde{q} = e^{2\pi i z},\ u \in \mathcal{O}^\times_{\mathbb{P}^1, P_1})$

"canonical coordinate" at $P_1 \in \mathbb{P}^1$.

$$\begin{tikzcd}
 & \mathbb{P}^1 \arrow[r, "\varphi"] & \Gamma\backslash D_\Sigma =: X \\
P_1 \in & U \arrow[u, phantom, sloped, "\subset"] \arrow[r] & V \arrow[u, phantom, sloped, "\subset"] \arrow[d] & \ni Q_1 \\
 & \tilde{q} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & q = e^{2\pi i(-a_1/a_3)} (= u\tilde{q}) \quad \mathbb{C} \ni q
\end{tikzcd}$$

$$\begin{tikzcd}
(M_{\mathbb{P}^1}/\mathcal{O}^\times_{\mathbb{P}^1})_{P_1} & (M_X/\mathcal{O}^\times_X)_{Q_1} \arrow[l, two heads, "\text{☺}"'] \\
 & (M_{\mathbb{C}}/\mathcal{O}^\times_{\mathbb{C}})_0 \arrow[u] \arrow[ul, "\sim"]
\end{tikzcd}$$

② At $P_\infty := \infty \in \mathbb{P}^1$. $\quad Q_\infty := \varphi(P_\infty) \in \Gamma\backslash D_\Sigma$

Then $\varphi^{-1}(Q_\infty) = \{P_\infty\}$.

Want to compute local ramification index of φ at P_∞.

Claim $(M_X/\mathcal{O}^\times_X)_{Q_\infty} \twoheadrightarrow (M_{\mathbb{P}^1}/\mathcal{O}^\times_{\mathbb{P}^1})_{P_\infty}$.

Similarly as ①, by the interpretation of "canonical coordinate" in [M93] into log language, we see the equality holds.

Other examples of 1 parameter mirror families

1. Mirror to weighted hypersurfaces

- $(2y_1 + y_2 + \cdots + y_5)^6 - 6^6\lambda\, y_1^2 y_2 \cdots y_5 = 0$
- $(4y_1 + y_2 + \cdots + y_5)^8 - 8^8\lambda\, y_1^4 y_2 \cdots y_5 = 0$
- $(5y_1 + 2y_2 + y_3 + \cdots + y_5)^{10} - 10^{10}\lambda\, y_1^5 y_2^2 y_3 y_4 y_5 = 0$

local monodromy [Klemm-Theisen 93]

generic Torelli [Shirakawa 09 prep] submitted

2. Borcea-Voisin type

F : elliptic curve with period $\exp\left(\frac{2\pi i}{6}\right)$

$(E_t)_{t\in\Delta}$: degenerating elliptic curve of Kodaira type I_m

$\left((F^2/(-1,-1) \times E_t)/((-1,1),-1)\right)_{t\in\Delta}$

The crepant resolution of this family has log Hodge structure passing through N_m-boundary.

[Green-Griffiths-Kerr 07]