Local Mirror Symmetry
and BPS state counting

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and work in progress with H. Awata

1. Introduction

Nekrasov's partition function of $SU(N)$ gauge theory:

$$Z_{\text{Nek}}(\epsilon_1, \epsilon_2, \vec{a}; q) = \sum_{k=0}^{\infty} q^k \sum_{\underline{Y} \in \mathcal{P}_N(k)} \frac{1}{\prod_{\alpha,\beta=1}^{N} n_{\alpha,\beta}^{\underline{Y}}(\epsilon_1, \epsilon_2, \vec{a})},$$

where $\underline{Y} = \{Y_\alpha\}_{\alpha=1}^{N}$ and

$$\begin{aligned} n_{\alpha,\beta}^{\underline{Y}}(\epsilon_1, \epsilon_2, \vec{a}) \; := \; & \prod_{s \in Y_\alpha} (-\ell_{Y_\beta}(s)\epsilon_1 + (a_{Y_\alpha}(s) + 1)\epsilon_2 + a_\beta - a_\alpha) \\ & \times \prod_{t \in Y_\beta} ((\ell_{Y_\alpha}(t) + 1)\epsilon_1 - a_{Y_\beta}(t)\epsilon_2 + a_\beta - a_\alpha), \end{aligned}$$

$\ell_Y(s)$: the leg length $\qquad a_Y(s)$: the arm length

$\mathcal{P}_N(k)$: the set of N-tuples of Young diagrams such that the total number of boxes is the instanton number k.

[N. Nekrasov, `hep-th/0206161`]

The contribution with the instanton number k is given by an equivariant integral of "1" over the moduli space $\mathcal{M}^{(N,k)}_{\text{ADHM}}$ of framed instantons on $\mathbb{C}^2$ with rank N and the second Chern class k, which is computed exactly by the localization theorem for the following toric action;

1. On $(z_1, z_2) \in \mathbb{C}^2$; $(z_1, z_2) \to (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2)$
2. The action of the maximal torus $(e^{ia_1}, \cdots, e^{ia_N}) \in T^{N-1}$ on $SU(N)$ with $\sum_{\ell=1}^{N} a_\ell = 0$.

In general the equivariant integration gives a Laurent series in the equivariant parameters $(\epsilon_1, \epsilon_2, \vec{a})$.

Localization Formula

If all the fixed points of G action on M, $(\dim M = 2\ell)$ are isolated, the equivariant integral of G-equivariantly closed form μ is given by

$$\int_M \mu = (-2\pi)^\ell \sum_{s \in M^G} \frac{\mu_0(s)}{\det^{\frac{1}{2}} \mathcal{L}_\xi(s)},$$

where $\mathcal{L}_\xi(s)$ is the homomorphism of $T_s M$ induced by the G action and μ_0 is the 0-form part of μ.

When $G = U(1)^r$, $\det^{\frac{1}{2}} \mathcal{L}_\xi(s) = \prod_{i=1}^{\ell} (k_i(s) \cdot \epsilon)$, where $(k_1(s), \ldots, k_\ell(s)) \in (\mathbb{Z}^r)^\ell$ are weights the toric action at s and ϵ is the generator of $\mathfrak{g}$.

By the localization formula Nekrasov computed $\int_{\mathcal{M}_{\mathrm{ADHM}}^{(N,k)}} 1$ = the "volume" of the moduli space, which arises from the path integral of the partition function of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory.

The fixed points of the toric action on $\mathcal{M}_{\mathrm{ADHM}}^{(N,k)}$ are in one-to-one correspondence with the elements of $\mathcal{P}_N(k)$.

[H. Nakajima : *Lectures on Hilbert schemes of points on surfaces*]

We only have to evaluate the weights (eigenvalues) of the toric action at each fixed point. Note that the denominator of Z_{Nek} has $\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\left(\sum_i \mu_{\alpha,i} + \sum_j \mu_{\beta,j}\right) = 2Nk = \dim_{\mathbb{C}} \mathcal{M}_{\mathrm{ADHM}}^{(N,k)}$ factors.

When $\epsilon_1 = -\epsilon_2 = \hbar$, we find

$$Z_{\mathrm{Nek}}(\hbar, \vec{a}; q) = \exp\left(-\sum_{r=0}^{\infty} \hbar^{2r-2} F_r(\vec{a}; q)\right)$$

and $F_0(\vec{a}; q) = \mathcal{F}_{SW}^{inst}(\vec{a}; q)$; the instanton part of the Seiberg-Witten prepotential.

This is regarded as a version of (local) mirror symmetry

A-side : $Z_{\mathrm{Nek}}(\hbar, \vec{a}; q)$ from the equivariant integral over the instanton moduli space by localization

B-side : $\mathcal{F}_{SW}^{inst}(\vec{a}; q)$ from the period integral of the Seiberg-Witten curve (to be explained in the next section)

[N. Nekrasov and A. Okounkov, hep-th/0306238]
[H. Nakajima and K. Yoshioka, math.AG/0306198]

2. Seiberg-Witten Prepotential

$\mathcal{N} = 2$ supersymmetric $SU(2)$ Yang-Mills theory
in four dimensions has $[A_\mu, \phi, +\ \mathrm{fermions}]$,
which are in the adjoint representation ($\mathfrak{su}(2)$-valued).

Potential for the complex scalar: $V(\phi) = \frac{1}{g^2}\mathrm{Tr}\ [\phi, \phi^\dagger]^2$
$\Rightarrow \phi = a\sigma_3$ to minimize $V(\phi)$.

Moduli of $\mathcal{N} = 2$ SUSY vacua: $u = \langle \mathrm{Tr}\ \phi^2 \rangle$
in the weak coupling (UV) region $|a| >> 1$, $u(a) \sim \frac{1}{2}a^2$.

Higgs mechanism
When $a \neq 0$, only $U(1)$ photon multiplet (proportional to σ_3) remains massless and other fields acquire masses with scale $|a|$.

Spontaneous symmetry braeking
Consequently the low energy effective symmetry is $U(1) \subset SU(2)$. We are interested in the non-perturbatively exact low energy effective action for the abelian gauge multiplet. The non-perturbative effects come from the existence of $SU(2)$ instantons (anti-self-dual connections).

By the restriction of $\mathcal{N} = 2$ supersymmetry the most general action of abelian gauge theory (up to two derivatives) is known to be determined by the holomorphic prepotential $\mathcal{F}(A)$ as follows;

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi}\text{Im}\left[\int d^4\theta \frac{\partial \mathcal{F}(A)}{\partial A}\overline{A} + \frac{1}{2}\int d^2\theta \frac{\partial^2 \mathcal{F}(A)}{\partial A^2} W^\alpha W_\alpha\right]$$

$\int d^4\theta, \int d^2\theta$ are integrals of superfields A, W_α.

$$\mathcal{F}_{\text{classical}}(a) = \frac{1}{2}\tau_0 a^2, \quad \tau_0 := \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

Special Kähler geometry (Why $\mathcal{F}$ is called "prepotential")

$$K(a,\bar{a}) = \text{Im}\,\frac{\partial \mathcal{F}}{\partial a}\bar{a} = \frac{i}{2}\left(\frac{\partial \overline{\mathcal{F}}}{\partial \bar{a}}a - \frac{\partial \mathcal{F}}{\partial a}\bar{a}\right) = \text{Im}\; a_D\,\bar{a}$$

gives a Kähler potential for the metric on the vacuum moduli space;

$$\begin{aligned}(ds)^2 &= \text{Im}\left(\frac{\partial^2 \mathcal{F}}{\partial a^2}\right) da d\bar{a} = \text{Im}\; da_D d\bar{a} \\ &= -\frac{i}{2}\left(\frac{da_D}{du}\frac{d\bar{a}}{d\bar{u}} - \frac{da}{du}\frac{d\bar{a}_D}{d\bar{u}}\right) du d\bar{u}\end{aligned}$$

The metric is invariant under $SL(2,\mathbb{R})$ action on (a, a_D) which realizes abelian electromagnetic duality.

Seiberg-Witten prepotential : $\mathcal{F}_{SW}(a,\Lambda)$
Exact (including instanton effects) prepotential has the following (instanton) expansion:

$$\mathcal{F}_{SW}(a,\Lambda) = \frac{\tau_0}{2}a^2 + \frac{a^2}{2}\left(\log\frac{a}{\Lambda} - \frac{3}{2}\right) + a^2 \sum_{k=1}^{\infty}\left(\frac{\Lambda}{a}\right)^{4k}\mathcal{F}_k$$

where Λ is the scale parameter in the renormalized theory. The second term is perturbative contribution (1-loop effect). The last part is the non-perturbative terms and the coefficients $\mathcal{F}_k$ are the "symplectic volume" $\mathcal{F}_k = \int_{\mathcal{M}_k}$ "1", where $\mathcal{M}_k$ is the moduli space of (framed) $SU(2)$ instantons on $\mathbb{R}^4$ with instanton number k.

Seiberg-Witten theory = "B-model" computation of $\mathcal{F}_{SW}$
[N. Seiberg and E. Witten, hep-th/9407087]

The prepotential $\mathcal{F}$ is a holomorphic section of an appropriate line bundle on the moduli space of vacua, which is $\mathbb{C}P^1$. Thus it can be reconstructed from monodromy data and global consistency.

Note also that a subgroup of $SL(2,\mathbb{R})$ is realized on the low energy effective theory, thus it is not surprising to find a close connection to the beautiful theory of automorphic functions.

The prepotential $\mathcal{F}_{SW}(a,\Lambda)$ is obtained by solving the Picard-Fuchs equation for the period integrals on $SU(2)$ Seiberg-Witten curve;

$$y^2 = (x^2 - u)^2 - 4\Lambda^4 ,$$

where u is the moduli parameter.
Consider the period integral

$$a(u) := \int_\alpha \lambda_{SW} , \qquad a_D(u) := \int_\beta \lambda_{SW}$$

of SW differential $\lambda_{SW} = -\frac{1}{\pi}\frac{x^2 dx}{y}$.

The period integrals satisfy the Picard-Fuchs equation

$$\mathcal{L}\vec{a} = 0, \qquad \mathcal{L} := \frac{\partial^2}{\partial u^2} - \frac{1}{4(\Lambda^4 - u^2)} ,$$

which has regular singularities at $u = \pm\Lambda, \infty$. The SW curve degenerates at $u = \pm 2\Lambda^2$, where a massless monopole (dyon) appears. This fact determines the monodromy of $(a(u), a_D(u))$ at the singularities and the solution is given by the hypergeometric functions;

$$\begin{aligned} a(u) &= \sqrt{2}\Lambda\alpha^{1/4}{}_2F_1(-\frac{1}{4},\frac{1}{4},1;\frac{1}{\alpha}), \quad \alpha = \frac{u^2}{\Lambda^4} \\ a_D(u) &= \frac{i\Lambda}{4}(\alpha-1){}_2F_1(\frac{3}{4},\frac{1}{4},2;1-\alpha) \end{aligned}$$

Substituting the inversion $u = u(a)$ (the inverse mirror map) of $a = a(u)$ to $a_D(u) = \frac{\partial \mathcal{F}_{SW}}{\partial a}$, we obtain a series expansion of $\mathcal{F}_{SW}(a, \Lambda)$ around infinity, which gives the coefficients of SW prepotential

$$\mathcal{F}_1 = \frac{1}{2^5}, \quad \mathcal{F}_2 = \frac{5}{2^{14}}, \quad \mathcal{F}_3 = \frac{3}{2^{18}}, \cdots\cdots$$

[reference] Expressions in terms of modular functions

$$\begin{aligned} j(\tau) &= \frac{(3\Lambda^4 + u^2)^3}{27(u^2 - \Lambda^4)^2}, \quad u(\tau) = \frac{\vartheta_3^4 + \vartheta_4^4}{\vartheta_2^4} \\ a(\tau) &= \frac{1}{3\vartheta_2^2}\left(E_2 + \vartheta_3^4 + \vartheta_4^4\right) \end{aligned}$$

3. Topological String and BPS states counting

The expansion of the Nekrasov's partition function

$$Z_{\text{Nek}}(\hbar, \vec{a}; q) = \exp\left(-\sum_{r=0}^{\infty} \hbar^{2r-2} F_r(\vec{a}; q)\right)$$

reminds us of a genus expansion in string theory. In fact the relation to the (topological) string theory becomes more transparent, if we consider the five dimensional ("trigonometric", or K theoretic) lift of $Z_{\text{Nek}}(\hbar, \vec{a}; q)$.

Five dimensional lift : $n^{Y}_{\alpha\beta}(\epsilon_1, \epsilon_2, a_\ell) \Rightarrow N^{Y}_{\alpha\beta}(q, t, Q_{\beta\alpha})$

$$N^{Y}_{\alpha\beta}(q, t, Q_{\beta\alpha}) = \prod_{s \in \mu_\alpha} \left(1 - t^{-\ell_{\mu_\beta}(s)} q^{-a_{\mu_\alpha}(s)-1} Q_{\beta\alpha}\right) \times \prod_{t \in \mu_\beta} \left(1 - t^{\ell_{\mu_\alpha}(t)+1} q^{a_{\mu_\beta}(t)} Q_{\beta\alpha}\right)$$

where $t := e^{-\epsilon_1}, q := e^{\epsilon_2}, Q_{\beta\alpha} = e^{a_\beta - a_\alpha}$.

We can show that, when $q = t = e^{-g_s}$, the five dimensional lift is nothing but the topological string amplitude on an appropriate local toric Calabi-Yau 3-fold K_S.

[A.Iqbal and A.-K. Kashani-Poor : hep-th/0212279,0306032]
[T. Eguchi and H.K. : hep-th/0310235]

For example for $SU(2)$ Yang-Mills theory the toric surface S is one of the Hirzebruch surfaces $\mathbb{F}_{0,1,2}$.
$\mathbb{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$ and $K_{\mathbb{F}_0}$ has two Kähler parameters t_B and t_F. The SW prepotential of $SU(2)$ theory is obtained by taking the "double scaling" limit $\epsilon \to 0$ of

$$Q_B = e^{-t_B} = (\epsilon\Lambda)^4 \ , \quad Q_F = e^{-t_F} = e^{-4\epsilon a} \ , \quad q = e^{-2\epsilon g_s}$$

in topological string amplitude on $K_{\mathbb{F}_0}$.

[S.Katz, A.Klemm and C.Vafa, hep-th/9609239]

In this limit the fiber $\mathbf{P}^1$ is collapsing, while the volume of the base $\mathbf{P}^1$ becomes quite large.

From the viewpoint of M theory, the free energy of the topological string is expected to have the following form;

$$F(\vec{t}, g_s) = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{r=0}^{\infty} \sum_{k=1}^{\infty} \frac{n_\beta^r}{k} \left(2 \sin \frac{k g_s}{2}\right)^{2r-2} e^{-k \cdot t_\beta}$$

where n_β^r is the integer invariants of Gopakumar-Vafa.

[R. Gopakumar and C. Vafa, hep-th/9812127]

The integrality of n_β^r comes from the fact that it is the multiplicities of BPS states in five dimensions! In the Calabi-Yau compactification of M theory, the BPS states with charge $\beta \in H_2(X, \mathbb{Z})$ arise from an $M2$-brane wrapping on a holomorphic cycle Σ with $[\Sigma] = \beta$.

The irreducible decomposition of the BPS states w.r.t. the five dimensional spin $SU(2)_L \times SU_R(2)$;

$$\left[(\frac{1}{2}, 0) \oplus 2(0,0)\right] \otimes N_\beta^{(j_L, j_R)} \left[(j_L, j_R)\right]$$

Since BPS states preserve half of $\mathcal{N} = 2$ SUSY, it always has the factor $(\frac{1}{2}, 0) \oplus 2(0,0)$.

Then the GV-invariants are identified by the relations;

$$N_\beta^{j_L} := \sum_{j_R} N_\beta^{(j_L, j_R)} (-1)^{2 j_R} (2 j_R + 1) \,,$$

$$\sum_r n_\beta^r (-1)^r (q^{1/2} - q^{-1/2})^{2r} = \sum_{j_L} N_\beta^{j_L} (q^{-j_L} + \cdots + q^{+j_L})$$

4. $SU(2)_L \times SU(2)_R$ spin decomposition

When $\epsilon_1 = -\epsilon_2 = g_s$, the 5-dimensional lift of Z_{Nek} reproduces topological string amplitude and related to the BPS state counting. A natural question is whether we can find a similar counting problem of BPS states even if $\epsilon_1 + \epsilon_2 \neq 0$. It is natural to expect that it gives a full information of $SU(2)_L \times SU(2)_R$ spin decomposition of the BPS states obtained from $M2$-branes wrapping over holomorphic 2-cycle β.

[T. Hollowood, A. Iqbal and C. Vafa, hep-th/0310272]

It is conjectured $SU(2)_L \times SU(2)_R$ spin decomposition of the five dimensional BPS states is mathematically identified as the Lefschetz decomposition of the cohomology of the moduli space of $D2$ branes.
Thus it is an interesting problem to check if the the 5-dimensional lift of Z_{Nek} gives a consistent "spectrum" with BPS state $\Leftrightarrow$ Cohomology class of the moduli space

The moduli space of $D2$-branes $\widetilde{\mathcal{M}}_\beta$ consists of the deformation of the holomorphic cycle in CY_3 together with the moduli of flat (=stable) $U(1)$ bundle over it.

We have a fibration $\pi : \widetilde{\mathcal{M}}_\beta \to \mathcal{M}_\beta$, where the base $\mathcal{M}_\beta$ is the moduli space of the two-cycle β without the choice of flat bundle. For example, if the two-cycle is generically the Riemann surface of genus g, then the generic fiber is T^{2g}, the Jacobian variety of the Riemann surface.

Both $\widetilde{\mathcal{M}}_\beta$ and $\mathcal{M}_\beta$ are Kähler manifolds and the Lefschetz action is defined by the multiplication of a Kähler form. It has been argued that the $SU(2)_L$ spin is identified with the Lefschetz decomposition along the fiber and the $SU(2)_R$ corresponds to the action on the base.

We thus have the following decomposition of the cohomology of the moduli space of $D2$ branes;

$$H^*(\widetilde{\mathcal{M}}_\beta) = \sum N_\beta^{(j_1,j_2)} \left[(j_1^{fiber}, j_2^{base}) \right]$$

In particular this identification implies that the $SU(2)_R$ spin contents with the highest $SU(2)_L$ spin is given by the Lefschetz decomposition of the cohomology of the base $\mathcal{M}_\beta$.

[H. Hosono, M.-H. Saito and A. Takahashi, hep-th/9901151, math.AG/0105148]

[S. Katz, A. Klemm and C. Vafa, hep-th/9910181]

The spectrum $N_\beta^{(j_L,j_R)}$ of the five dimensional BPS states contributes to the low energy effective action as follows;

$$\begin{aligned} &F(q,t;Q_\beta) \\ &= \sum_{\beta\in H_2(X,\mathbb{Z})} \sum_{j_L,j_R} \sum_{n=1}^{\infty} \frac{N_\beta^{(j_L,j_R)}}{n(q^{n/2}-q^{-n/2})(t^{n/2}-t^{-n/2})} \\ &\times \left(u^{-n\cdot j_L}+\cdots+u^{n\cdot j_L}\right)\left(v^{-n\cdot j_R}+\cdots+v^{n\cdot j_R}\right) Q_\beta^n . \end{aligned}$$

where $u := qt, v := q/t$ and $Q_\beta = e^{-t_\beta}$.

Let us look at the Nekrasov's formula for pure $SU(2)$ case, which corresponds to local $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$.
Take a holomorphic curve $\beta \subset \mathbf{F}_0$ with bidegree (a,b).
The curve has generically genus $g = (a-1)(b-1)$.

The result for the class $B+nF$ with genus zero is obtained from the one instanton part; ($Q := e^{a_1-a_2}$)

$$\begin{aligned} &F^{one\ inst}(q,t;Q) \\ &= \frac{vQ}{(q^{1/2}-q^{-1/2})(t^{1/2}-t^{-1/2})} \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(v^{k+\frac{1}{2}}+v^{-k-\frac{1}{2}})\right) Q^n \end{aligned}$$

Comparing with the general structure of the free energy $F(q,t;Q_\beta)$, we find

$$N_{B+nF}^{(j_L,j_R)} = \delta_{j_L,0}\,\delta_{j_R,n+\frac{1}{2}}$$

This result is consistent with the geometry of the moduli space $\widetilde{\mathcal{M}}_{B+nF}$.

Firstly, the curve $B+nF$ has generically genus zero and the fiber (the moduli of flat $U(1)$ bundle) is trivial, implying the left spin vanishes. $\Rightarrow \delta_{j_L,0}$

Furthermore the moduli space of curves of bi-degree (a,b) in $\mathbf{P}^1 \times \mathbf{P}^1$ (without the flat budle over them) is shown to be $\mathbf{P}^{(a+1)(b+1)-1}$. When $(a,b) = (1,n)$, the moduli space is $\mathbf{P}^{2n+1}$ and the Lefschetz decomposition gives a single multiplet of spin $n+\frac{1}{2}$. $\Rightarrow \delta_{j_R,n+\frac{1}{2}}$

Similarly by looking at the two instanton part of the free energy, we obtain the following "spectrum" of BPS states arising from the homology class $2B+kF$;

$$\bigoplus_{(j_L,j_R)} N_{2B+kF}^{(j_L,j_R)} (j_L,j_R) = \bigoplus_{\ell=1}^{k} \bigoplus_{m=1}^{k-\ell+1} \left[\frac{m+1}{2}\right] \left(\frac{\ell-1}{2}, \frac{3\ell+2m}{2}\right) .$$

For lower values of k;

$$k=1 \ : \ (0,\frac{5}{2})$$

$$k=2 \ : \ (\frac{1}{2},4)\oplus(0,\frac{7}{2})\oplus(0,\frac{5}{2})$$

$$k=3 \ : \ (1,\frac{11}{2})\oplus(\frac{1}{2},5)\oplus(\frac{1}{2},4)\oplus 2(0,\frac{9}{2})\oplus(0,\frac{7}{2})\oplus(0,\frac{5}{2})$$

$$k=4 \ : \ (\frac{3}{2},7)\oplus(1,\frac{13}{2})\oplus(1,\frac{11}{2})\oplus 2(\frac{1}{2},6)\oplus(\frac{1}{2},5)\oplus(\frac{1}{2},4) \oplus 2(0,\frac{11}{2})\oplus 2(0,\frac{9}{2})\oplus(0,\frac{7}{2})\oplus(0,\frac{5}{2})$$

The spin content of the highest left spin ($\ell = k, m = 1$) is $\left(\frac{k-1}{2}, \frac{3k+2}{2}\right)$. This is consistent with the geometry of the moduli space of $D2$ branes as follows;

The genus of the curves with bi-degree $(2, k)$ is generically $k-1$ and the generic fiber is T^{2k-2}, whose Lefschetz decomposition is $\left[(\frac{1}{2}) \oplus 2(0)\right]^{\otimes(k-1)}$. Hence, the highest left spin is $(k-1)/2$.
The right spin contents with the highest left spin agree with the fact that it is identified with the Lefscetz decomposition of $\mathcal{M}_{2B+kF} = \mathbf{P}^{3k+2}$.

Conjectures and Challenges

1. Free energy of 5 dimensional lift of Z_{Nek} is expanded in terms of the characters of $SU(2)_L \times SU(2)_R$;

$$F(q,t) \sim \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{(j_L,j_R)} N_\beta^{(j_L,j_R)} \chi_{j_L}(qt) \cdot \chi_{j_R}(q/t)$$

2. The coefficients $N_\beta^{(j_L,j_R)}$ are integral
3. The expansion gives the the Lefschetz decomposition of the cohomology $H^*(\widetilde{\mathcal{M}_\beta})$ of the moduli space of D branes;