A theory of Infinitely Near singularities

by means of

Idealistic Exponents

a "global-local technique" conceptually bridging
global problems with local problems

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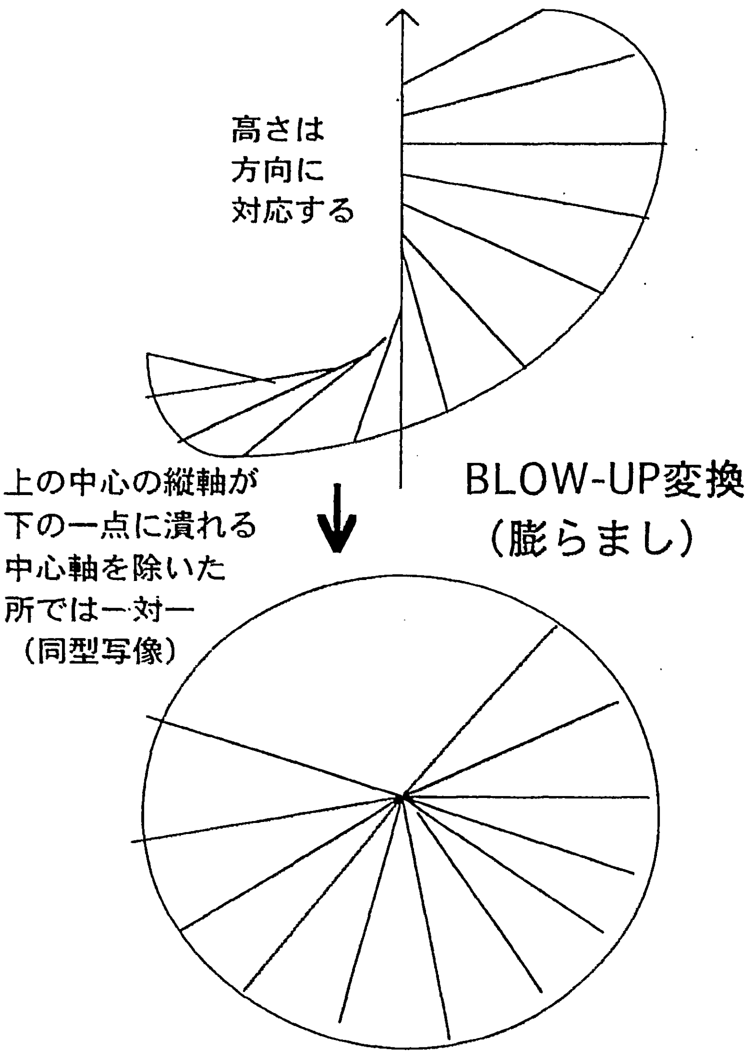
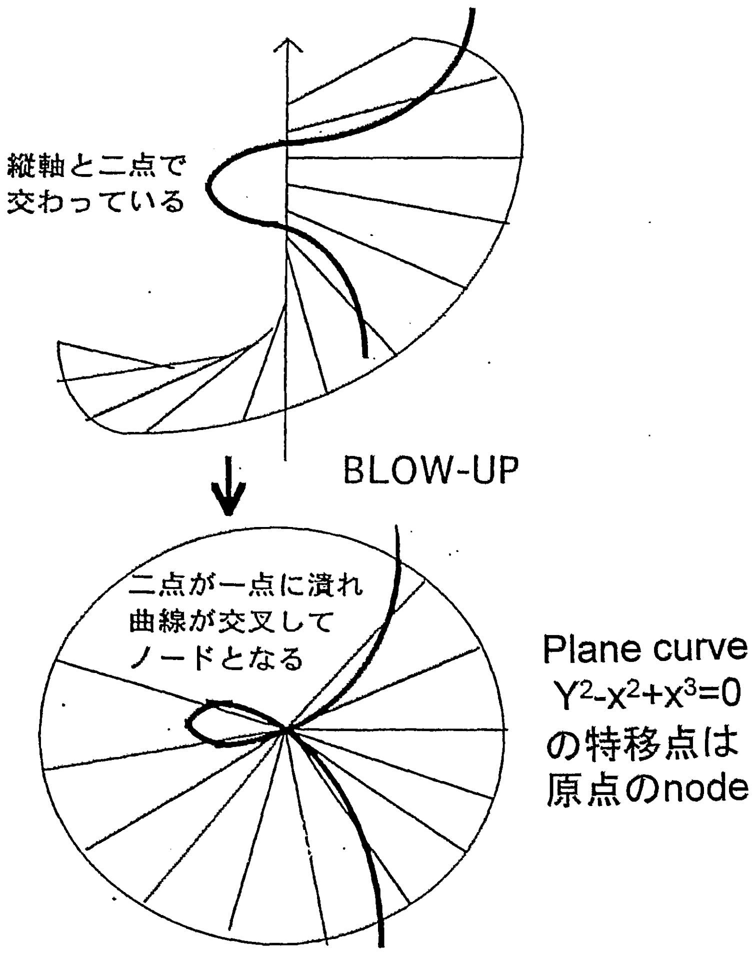
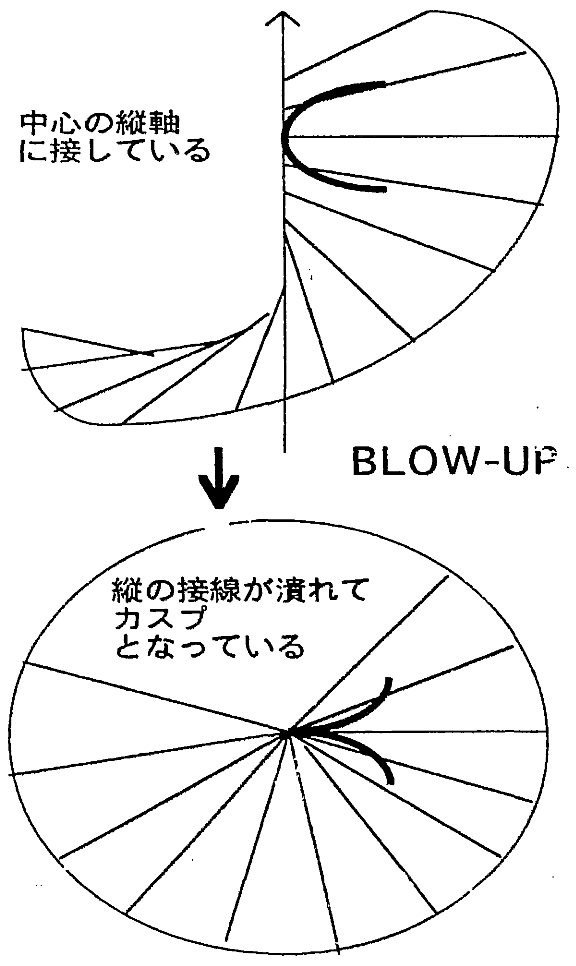
Infinitely near singularities
を観るためには
顕微鏡に相当するものとして
Blowing-up
を用いる。


高さは
方向に
対応する
上の中心の縦軸が
下の一点に潰れる
中心軸を除いた
所では一対一
（同型写像）
BLOW-UP変換
（膨らまし）


縦軸と二点で
交わっている
BLOW-UP
二点が一点に潰れ
曲線が交叉して
ノードとなる
Plane curve
$Y^2-x^2+x^3=0$
の特移点は
原点のnode



Plane curve
$Y^2 - x^3 = 0$
の特異点は
原点のcusp

例： $1<b<a, (a,b)=1$

Plane curve defined by $y^b - x^a = 0$

Blow up at the origin (0,0)

Transform defined by $y_1{}^b - x_1{}^{a-b} = 0$

Blow up at the origin (0,0)

Transform defined by $y_2{}^b - x_1{}^{a-2b} = 0$

Blow up at the origin (0,0)

できるだけ」繰り返す。すると整数部分 [a/b] 回まで Multiplicity b が続く。したがって、

Infinitely near singular points が [a/b] 個あると言う。

さて、さて、
一般論の はじまり、 はじまり！

有理数のイデアル化

出発点の発想：　　整数　$a \mapsto$　イデアル　(?)?[x]

有理数：　対　(a,b), $b>0$, (a,b は整数) の同値を次のように定義する：

$$(a,b) \sim (a',\ b') \Longleftrightarrow \exists\, c,\ c'>0 \text{ such that } ac' = a'c, bc' = b'c$$

イデアル化：　対　(J,b), $b>0$, (J: 多様体上のイデアル) の同値を次のように定義する：

$$(J,b) \sim (J',\ b') \Longleftrightarrow \exists\, c,\ c'>0 \text{ such that } J^{c'} = J'^{c}, bc' = b'c$$

有理数：　$\min\{(a,b), (a',b')\} = (\min\{ab', a'b\}, bb')$

イデアル化：　$(J,b) \cap (J',b') = (J^{b'} + J'^{b}, bb')$

有理数：　$(a,b) \le (a',b') \Longleftrightarrow ab' \le a'b$

イデアル化：　$(J,b) \subset (J',b') \Longleftrightarrow J^{b'} \supset J'^{b}$

かくて、(J,b) を idealistic exponent と呼ぶ。

Idealistic exponent
とは
pair (J, b)、ただし
b : 整数 ＞ 0
J : 多様体上の
イデアルの層

Our ambient scheme Z will be smooth of finite type over a perfect field k. We study the *infinitely near* singularity of an *idealistic exponent* on Z, say $E = (J, b)$, with a coherent ideal J on Z and an integer $b > 0$.

Example.
$Z = Spec(k[x]), x = (x_1, \cdots, x_n)$. $H = Spec(k[x]/fk[x])$ with $f \in k[x]$. Consider the idealistic exponent $E = (J, b)$ with $J = f\mathcal{O}_Z$ and $b = max\{mult_\xi(H), \xi \in H\}$. We aim to reduce the multiplicities to $< b$.

The infinitely near singularity of

$$E = (J, b)$$

is represented by those *blowing-up diagrams*, called LSB, which are *permissible* for it. So we recall the notion of LSB and *permissibility*.

We will use an arbitary finite system of indeterminates $t = (t_1, t_2, \cdots, t_a)$. We let $Z[t] = Z \times_k Spec(k[t])$ and $E[t] = (J[t], b)$ with $J[t] = J\mathcal{O}_Z[t]$.

Definition

A *local sequence of smooth blowing-ups*, called *LSB over Z* for short, means:

$$\begin{array}{ccccc} Z_r & \overset{\pi_{r-1}}{\rightarrow} & U_{r-1} \subset Z_{r-1} & \overset{\pi_{r-2}}{\rightarrow} & \\ & & \bigcup & & \\ & & D_{r-1} & & \\ & & & \cdots & \end{array}$$

$$\begin{array}{ccccc} \overset{\pi_1}{\rightarrow} & U_1 \subset Z_1 & \overset{\pi_0}{\rightarrow} & U_0 \subset Z_0 = Z \\ & \bigcup & & \bigcup \\ & D_1 & & D_0 \end{array}$$

where U_i is an open subscheme of Z_i, D_i is a regular closed subscheme of U_i and the arrows mean: $\pi_i : Z_{i+1} \rightarrow U_i \subset Z_i$ is the blowing-up with center D_i.

The *permissibility* of an LSB for $E = (J, b)$ is defined inductively. It then is enough to define *permissibility* of a blowing-up and the *transform* of E by such a blowing-up. For an open $U_0 \subset Z$, E is replaced by its restriction $E \,|\, U_0 = (J \,|U_0, b)$. Define

$$ord_\eta(E) = ord_\eta(J)/b$$

and

$$\begin{aligned} Sing(E) &= \{\eta \in Z \,|\, ord_\eta(E) \geq 1\} \\ &= \{\eta \in Z \,|\, ord_\eta(J) \geq b\} \end{aligned}$$

which is called *singular locus* of E.

A blowing-up $\pi : Z_1 \to Z$ with center D is defined to be ***permissible*** for E if D is smooth and $D \subset Sing(E)$. The ***transform*** $E_1 = (J_1, b)$ of E is defined by

$$J_1 P^b = J\mathcal{O}_{Z_1}$$

with respect to π, where

$$P = \big(ideal(D \subset Z)\big)\mathcal{O}_{Z_1}$$

called the exceptional divisor. Note that P is nonzero locally principal so that J_1 is uniquely determined as an ideal sheaf in $\mathcal{O}_{Z_1}$. Now the notion of ***permissibility*** is defined for LSB.

An idealistic exponent (J,b)
should be dealt with
as a symbol which represents:

the family of the t-indexed sets
of all permissible LSB's
over Z[t]
where the index t is
any finite system of indeterminates

For a pair of idealistic exponents $E_i = (J_i, b_i), i = 1, 2$, we define the *inclusion*:

$$E_1 \subset E_2$$

to mean the condition: *Pick any $t = (t_1, \cdots, t_a)$. If an LSB over $Z[t]$ is permissible for $E_1[t]$, so is it for $E_2[t]$.*

Equivalence: $E_1 \sim E_2$ will mean that $E_1 \subset E_2$ and $E_1 \supset E_2$.

Equivalence: $\cap_j E_j \sim E$ will mean that, $\forall t$, an LSB over $Z[t]$ is permissible for $E_j[t]$ for all j if and only if it is so for $E[t]$.

[1] $(J^e, eb) \sim (J, b)$ for every $e > 0$.

[2] If $b_i \mid m, i = 1, 2$,

$$(J_1, b_1) \cap (J_2, b_2) \sim (J_1^{\frac{m}{b_1}} + J_2^{\frac{m}{b_2}}, m)$$

[3] $(J_1 J_2, b_1 + b_2) \supset (J_1, b_1) \cap (J_2, b_2)$ If $Sing(J_i, b_i + 1) = \emptyset$ for both $i = 1, 2$, then they are *equivalent* .

[4] Compare $F_1 = (J, b_1)$ and $F_2 = (J, b_2)$ with $b_1 > b_2$. Then we have

1) $F_1 \subset F_2$.

2) Their final transforms by an LSB differ only by a power product of the ideals of exceptional divisors.

[5] $(\bar{J}, b) \sim (J, b)$ for the *integral closure* $\bar{J}$ of J.

Three Technical Key Theorems

Numerical Exponent Theorem
Differential Theorem
Ambient Reduction Theorem

For an idealistic exponent $E = (J, b)$ on an ambient scheme Z, we define:
$ord_\xi(E) = ord_\xi(J)/b$ for $\xi \in Z$.

Numerical Exponent Theorem

Let $E_i = (J_i, b_i), i = 1, 2,$ be two idealistic exponents on an excellent Z.

If we have $(E_1) \subset (E_2)$ then we have $ord_\xi(E_1) \leq ord_\xi(E_2)$ for $\forall \xi \in Z$.

In particular, if we have $E_1 \sim E_2$ then $ord_\xi(E_1) \leq ord_\xi(E_2)$ for $\forall \xi \in Z$.

Diff Theorem

If $\mathcal{D}$ is any left
O_Z-submodule of $Diff_Z^{(i)}$
then we have the follwoing inclusion
in the sense of idealistic exponents

$$(\mathcal{D}J, b-i) \supset (J, b)$$

or equivalently

$$(J, b) \cap (\mathcal{D}J, b-i) \sim (J, b)$$

Ambient Reduction Theorem

Let $W \subset Z$ be closed smooth over a perfect field.

Theorem

$$\bigcap_{0 \leq j < b} \left((Diff_Z^{(j)} J) O_W, b - j \right)$$

is an *ambient reduction of* (J, b) *from* Z *to* W. This means:

Pick any indeterminates $t = (t_1, \cdots, t_r)$ *and any LSB on* $Z[t]$, *such that the strict transforms of* $W[t]$ *contain all its centers. Then the LSB permissible for* $E[t]$ *if and only if so is the induced LSB on* $W[t]$ *for* $F[t]$.

Given $E = (J, b)$ on Z, let $J_{max}(a)$ be the maximal one among all those ideal I such that $(I, a) \supset (J, b)$. We then define a graded $\mathcal{O}_Z$-algebra

$$\wp(E) = \sum_{0 \le a < \infty} J_{max}(a) T^a$$

where T is an auxiliary indeterminate whose powers corresponds to degrees. We call $\wp(E)$ the *characteristic algebra* of E, which generalizes the classical *first characteristic exponent* of a plane curve singularity.

Given $E = (J, b)$ on Z, define

$$E^{\natural} = (J^{\natural}, b^{\natural})$$

where $b^{\natural} = b!$ and

$$J^{\natural} = \sum_{0 \leq \mu \leq b-1} \left(Diff^{(\mu)}_{Z/\mathbf{k}} J \right)^{b^{\natural}/(b-\mu)}$$

Theorem

$\wp(E)$ is the integral closure of

$$\sum_{\alpha=0}^{\infty} (J^{\natural})^{\alpha} T^{b^{\natural}\alpha} \quad \subset \quad \sum_{\alpha=0}^{\infty} \mathcal{O}_Z T^{\beta}$$

Hence $\wp(E)$ is finitely presented as $\mathcal{O}_Z$-algebra.

(H.Hironaka, J. of Korean Math.Soc. 40(2003),901-920)

基礎体が標数ゼロのときは、特異点解消は次のような要領で簡単に解決できる。

総ての次元で証明しようとすると当然ながら帰納法を用いるのがよい。（帰納法を嫌う人は大いに頑張って帰納法の証明手順からinvariantsを引き出せばよい。）

To make the idea clearer, take the case of

a hypersurface $X \subset \mathbb{C}^n$

defined by $f(x_1, \cdots, x_n) = 0$.

Incidentally, we have a technique to reduce the general case to hypersurface case. (*Idealistic exponents of singularity*, J.Hopkins U.Press,(1977)pp.52-125.

We use differential operators

$$\partial_\alpha = \frac{\partial^{|\alpha|}}{\partial x_1{}^{\alpha_1} \partial x_2{}^{\alpha_2} \cdots \partial x_n{}^{\alpha_n}}, \quad \forall \alpha \in \mathbb{Z}_0^n$$

We look into the singularity of X at a point ξ, say $\xi = (0)$ and $f(\xi) = 0$. Let $b = mult_\xi(X) = ord_\xi(f)$, and consider the set

$$\{ \partial_\alpha(f) \mid \alpha \in \mathbb{Z}_0^n, |\alpha| = b - 1 \}$$

out of which we choose a maximal system of those having linearly independent linear terms:

$$\{ f_j' = \partial_{\alpha_j}(f), 1 \leq j \leq d \}$$

This d is called *Zariski codimension* of X at ξ.

By the implicit function theorem, we obtain a smooth submanifold W of codimenion d (locally at ξ) in $\mathbb{C}^n$, defined by $f'_1 = \cdots = f'_d = 0$. This W has the **maximal contact** with X in $\mathbb{C}^n$ at ξ.

Consider the ideal in the ambient space $Z = \mathbb{C}^n$:

$$J = \sum_{0 \leq i \leq b-1} \left(Diff_Z^{(i)} f \right)^{\frac{b!}{b-i}}$$

and then let $\bar{J}$ be the ideal in W induced by J.

Then the problem of reducing the multiplicity of f on Z to less than b everywhere (by means of successive "permissible" blowing-ups) is completely *equivalent* to the reducing the multiplicity of the ideal $\bar{J}$ on W (by similar means) to less than $b!$ everywhere.

Thus, the problem in dimension n is reduced to a problem in dimension $n - d < n$.

これで総ての次元の多様体に対する特異点解消の証明は「ほぼ」終了。何故なら一次元の場合は大抵の条件のもとで特異点解消は簡単に出来るから。

この「ほぼ」だが、証明は９９％正しい、１００％ではないと言うこと。

騙されてはいかん。

しかし、残された１％は真面目に数学に通じたひとであれば、そして辛抱強いひとであれば誰でも完璧にまで修正できる程度。

疑問１. 始めに一つの方程式　f (equation　for the　hypersurface) から出発して、低次元の連立方程式の場合に帰着した。これ如何に？

（答え） No problem！ 最初から idealistic exponent (J,b) で始めておけばよい。(一つの　f　にかわりに関数のシステムで生成されたイデアル　J　で置き換えるだけのこと。）

疑問 2. 始めに b は multiplicity の最大値とした。だから W を定義する方程式は empty でないし、 dim W < dimZ. これが = だったら帰納法は全く駄目。

（答え） Very little problem！ 一般に （J, b） と （J, b'）, b' < b, との違いは normal crossings (accumulated exceptional divisors) だけ。 それは少し知恵を働かせれば簡単に処置できる。

基礎体の標数が　p>0　の場合は？

問題が一段と複雑になることが知られている。その場合、大局的問題よりも、局所的問題に関して、自然で無理押しでない有効な方法が模索されている。

まず、　p>0　の場合の局所的な研究への門出の第一歩がある。

Local theory: Initial decomposition

Pick a closed point $\xi \in Z$ and let $R = \mathcal{O}_{Z,\xi}$. Then $\kappa = R/M$ with $M = max(R)$ is perfect. Given $f \in R$ with $ord_M(f) = m > 0$, the initial form $in_M(f)$ is a nonzero homogeneous polynomial in $\bar{R} = \kappa[M/M^2]$.

Lemma

$\exists$ a base $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n)$ of M/M^2 such that $\exists$ $r, 1 \leq r \leq n$, and $q_i = p^{e_i}, e_i \geq 0, 1 \leq i \leq r$, such that

1. $1 \leq q_1 \leq q_2 \leq \cdots \leq q_r \leq m$
2. $in_M(f) \in \kappa[\bar{x}_1^{q_1}, \cdots, \bar{x}_r^{q_r}]$
3. $\forall i, \exists D_i \in Diff_{R/k}^{(m-q_i)}, in_M(D_i f) = \bar{x}_i^{q_i}$.

It follows that $\exists$ a weighted homogeneous $F(Y) \in R[Y], Y = (Y_1, \cdots, Y_r)$, with $wt(Y_i) = q_i$, such that $wtdeg(F) = m$ and

$$f - F(D_1 f, \cdots, D_r f) \in M^{m+1}.$$

Theorem

Given $E = (J, b)$ on Z, there exists an equivalence

$$E \sim \left(\cap_{i=1}^{r} E_i\right) \cap F$$

where $E_i = (fR, q_i), 1 \leq i \leq r$, and $F = (I, a)$ with $ord_M(I) > a$.

Inductively, the problem can be reduced to the case of

$$E \sim (\cap_{i=1}^{r} E_i) \cap F$$

where

1. $F = (I, a)$ where I is a normal crossing ideal of only exceptional divisors
2. $E_i = (f_i R, q_i)$ with

$$f_i = x_i^{q_i} + \mathbf{f_{i0}} + \sum_{j=1}^{q_i - 1} f_{ij} x_i^j$$

where $q_i = p^{e_i}$ and $0 \leq e_1 \cdots \leq e_r$.

3.

$$f_{ij}^a \in I^{q_i - j},$$

where $\forall (i, j), 1 \leq i \leq r, 1 \leq j \leq q_i - 1$

かくて、数の列

（$n, r, q_1, q_2, \ldots\ldots, q_r$）

に関する

帰納法

（Inductive argument）

が始まる？

本当に重大なステップ(local)はまだまだ。

はじまり！

はじまり！　で

本日は　おわり