代数トポロジーと場の理論

山下真由子 (RIMS)

自己紹介

- 院生の頃は 作用素環論 の研究室に所属。(~2019)
  (ᶜ解析学)
  (指導教官: 河東泰之先生 (東大))

作用素環の K理論、"非可換幾何学" をやっていた。

代数トポロジーの道具。　作用素環を幾何学に応用する分野。

特に Atiyah-Singer の指数定理 (の"非可換版") に興味があった。

- 幾何・トポロジーの超重要定理。
- (最近) 数理物理学で重要になってきている。

⇝ 理論物理学者との交流・共同研究をするように。(M2〜)

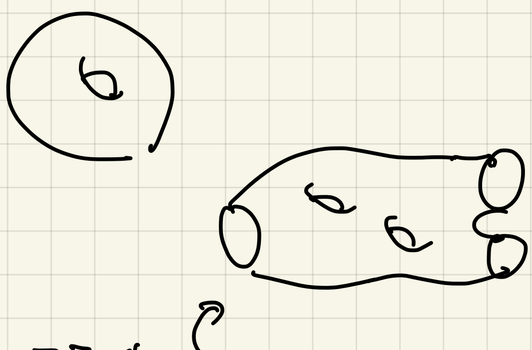
- 現在は、K理論に限らず 代数トポロジー と その数理物理との関連 に興味をもって研究している。

代数トポロジー　(数学)

幾何学的オブジェクト(位相空間、多様体…)
の情報を**代数的な情報**におとして研究する.

X : 位相空間

M : 多様体



群作用 G

⇒

常(コ)ホモロジー

$H^*(X;\mathbb{Z})$, $H_*(X;\mathbb{Z})$

$H^*_{dR}(M;\mathbb{R})$ de Rham コホモロジー

$= \Lambda^*_{clo}(M)/d\Lambda^*(M)$

$K^*(X)$ K-理論

一般(コ)ホモロジー理論

$E^*(X)$, $E_*(X)$

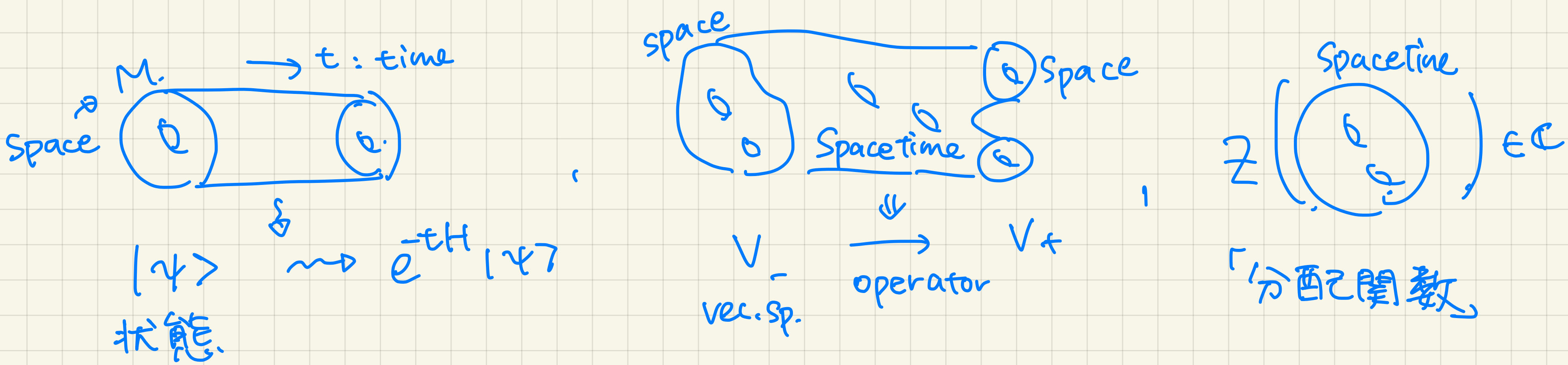
$E^*_G(X)$ 同変コホモロジー

スペクトラム、ホモトピー論、…

(量子)場の理論, QFT　(物理)

Quantum Field Theory

理論物理の一分野



数学からも古くから興味をもたれてきた.

数学的に理解できそう! しかしなかなかできない…

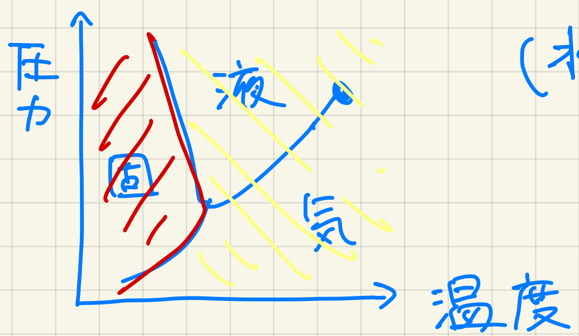
⇝ さまざまな数学を生み出してきた.

ex. Chern-Simons 理論 ⇝ 結び目の不変量 …

Donaldson 理論,
Siberg-Witten 理論 ⇝ 4次元トポロジー …

Today's topic : 場の理論に代数トポロジーを応用しよう!

・近年、QFTの分類に代数トポロジーが有用だと分かってきて、注目を集めている。



(水の三相)

例 Hamiltonianの分類 ⟵ K-理論, KO-理論

$H : L^2(\mathbb{Z}^n) \to L^2(\mathbb{Z}^n)$ $K^*(T^n)$, $KO^*(T^n)$

$K_G^*(X)$, …

・一般的な「予想」(期待?) (Kitaev, Freed-Hopkins, …)

「"Invertible"なQFTは一般コホモロジーで分類される。」

⟷ 量子異常 (アノマリー)

私の研究 のいくつか、

- 一般コホモロジーは抽象的… そこで

物理的な（QFTと直接結びつく）定式化を
与えよう！

  - $I\Omega^{G}$ (Anderson dual to G-bordism) という一般コホモロジー に対して、invertible QFTを抽象化したモデルを与えた。
（米倉 - Y, 2021）

⇝（知られているQFT）$\mapsto$ $(I\Omega^{G})^{*}(X)$ の元.

- 「アノマリー ↔ 一般コホモロジー」の対応を用いて、
数学の議論によってアノマリーの解析を行おう！

  - $TMF \to I\Omega^{String}$ というコホモロジーの変換を
調べることで、heterotic弦理論のアノマリーがないことを
示した。（立川 - Y, 2021）

目次

§1. Generalized (Co)homology Theories.

A generalized cohomology theory $\{E^n\}_{n\in\mathbb{Z}}$:

- X : top. sp (CW complex) $\longmapsto$ $E^n(X)$: Abelian grp.
- Contravariant. : $f: X \to Y$ continuous $\mapsto$ $f^*: E^n(Y) \to E^n(X)$
- Eilenberg - Steenrod Axioms.
  - Homotopy invariance, Exactness. Exicion, $\cdots$

A generalized Homology theory $\{E_n\}_{n\in\mathbb{Z}}$:

- X $\longmapsto$ $E_n(X)$
- Covariant : $f_*: E_n(X) \to E_n(Y)$ +Axioms.

There can be various "models" for a single $\{E^n\}_n$: (realizations)

Example ① Ordinary (Co) Homology $H^*(-;A)$ $H_*(-;A)$

A: Abelian grp. ($A = \mathbb{Z}, \mathbb{R}, \cdots$)

- **Singular model**: $C_n(X;A) = A[\mathrm{Map}: \Delta^n \to X]$

$$\xrightarrow{\partial} C_{n+1}(X;A) \xrightarrow{\partial_{n+1}} C_n(X;A) \xrightarrow{\partial_n} \cdots$$ singular chain cpx

$$H_n^{sing}(X;A) := \mathrm{Ker}(\partial_n) / \mathrm{Im}(\partial_{n+1})$$

$$C^n(X;A) = \mathrm{Hom}(C_n(X;\mathbb{Z}), A)$$

$$\xrightarrow{\delta^{n+1}} C^{n-1}(X,A) \xrightarrow{\delta^{n-1}} C^n(X;A) \xrightarrow{\delta^n} \cdots$$ cochain cpx

$$H^n_{sing}(X;A) := \mathrm{Ker}(\delta^n) / \mathrm{Im}(\delta^{n-1})$$

- **de Rham model**: M: manifold $\Lambda^n(M)$: differential n-forms.

$$\xrightarrow{d} \Lambda^{n-1}(M) \xrightarrow{d} \Lambda^n(M) \xrightarrow{d} \cdots$$ de Rham complex

$$H^n_{dR}(M) := \mathrm{Ker}(d) / \mathrm{Im}(d)$$

Example ② K-theory, KO-theory

- Vector bundle model : X : compact

$K^0(X) :=$ the Grothendieck group of $\mathbb{C}$-vector bundles over X.

i.e. $K^0(X) \ni [V]-[W]$ formal difference

homotopy class of $\begin{array}{c} V \\ \downarrow \mathbb{C}^n \\ X \end{array}$, $\begin{array}{c} W \\ \downarrow \mathbb{C}^m \\ X \end{array}$ $\mathbb{C}$-vec. bdles.

$KO^0(X) :=$ " $\mathbb{R}$-vector bundles. "

- Fredholm operator model

$\mathcal{H}_{\mathbb{C}}$: ∞-dim. Separable $\mathbb{C}$-Hilbert space. (e.g. $\mathcal{H} = \ell^2(\mathbb{Z})$)

$D : \mathcal{H} \to \mathcal{H}$ bounded $\mathbb{C}$-linear op. is Fredholm

$\overset{\text{def}}{\Longleftrightarrow}$ $\underline{\mathrm{Index}(D)} = \dim \ker(D) - \dim \mathrm{Coker}(D) < \infty$.

$$K^0(X) := \left\{ \{D_x\}_{x\in X} \;\middle|\; \begin{array}{l} \text{conti. family of} \\ \text{Fredholm operators} \end{array} \right\} \Big/ \sim_{\text{homotopy}}.$$

※ related to Atiyah-Singer's <u>Index</u> Theorem.

Example ③ G-bordism Homology Theory $\Omega_*^G(-)$

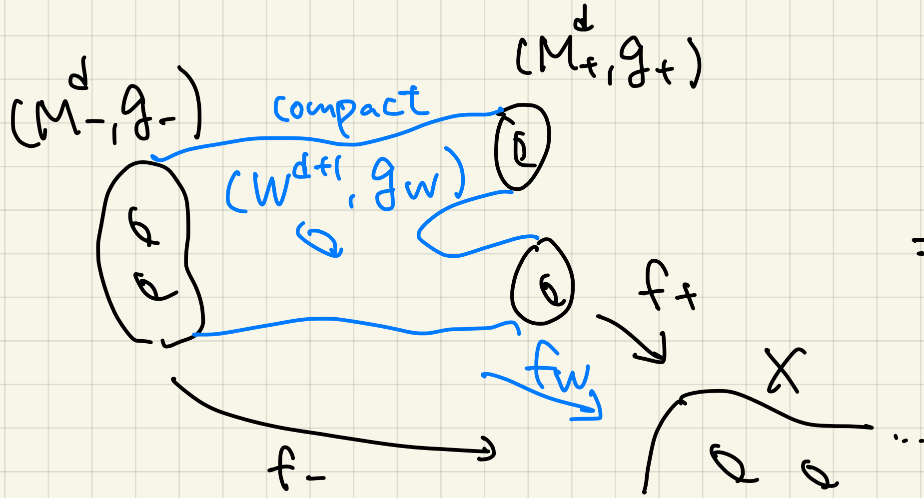
Fix G : structure group on manifolds.

Ex. $G = SO$ (oriented), Spin, 1 (framed), U (almost complex)

$$\left(G = \{G_d, \rho_d, s_d\}_{d \geq 0}, \quad \begin{tikzcd} G_d \arrow[r, "s_d"] \arrow[d, "\rho_d"'] \arrow[dr, phantom, "\circlearrowleft"] & O(d, \mathbb{R}) \arrow[d] \\ G_{d+1} \arrow[r, "s_{d+1}"'] & O(d+1, \mathbb{R}) \end{tikzcd} \right)$$

X: space, $\Omega_d^G(X)$: d-dim stable G-bordism group.

$$\Omega_d^G(X) := \left\{ (M^d, g, f) \;\middle|\; \begin{array}{l} M: \text{d-dim. closed manifold} \\ g: \text{stable } G\text{-str. on } TM \\ f: M \to X \text{ conti} \end{array} \right\} \Big/ \underset{\text{bordism}}{\sim}$$



$\Rightarrow [M_-, g_-, f_-] = [M_+, g_+, f_+]$ in $\Omega_d^G(X)$

§2. QFT (Quantum Field Theory, 場の理論)

"Theories" in physics: various formulations

- Hamiltonian picture,
- Lagrangian picture,
- **Cobordism picture.** ← Today's focus.

(d-1, d)-dim QFT is a "physically meaningful" symmetric monoidal **Functor**, ← No axiom yet.

$$I : \mathrm{Bord}^{S}_{(d-1,d)} \to \mathrm{Vect}_{\mathbb{C}}$$

$\mathrm{Bord}^{S}_{(d-1,d)}$ ← Bordism Category

$\mathrm{Vect}_{\mathbb{C}}$ ← Cat. of $\mathbb{C}$-vector spaces.

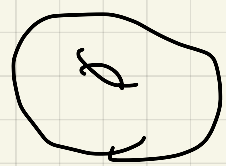
S: "structure" on manifolds.

$\mathrm{Bord}^{S}_{(d-1,d)}$: Bordism category

S: "structure"

- orientation, Spin Structure : Tangential structure.
  SO $\quad$ Spin.
- Principal H-bundle (H: fixed Lie group)
- map to X. (X : a fixed manifold)
- connection , metrics .

Object : (d-1)-dim closed manifold with a structure

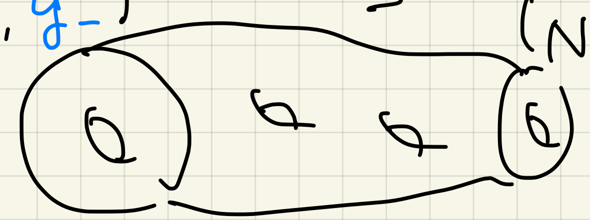


(N^{d-1}, g)

morphism: d-dim compact manifold with boundary

(M^d, g_W) with a Structure.

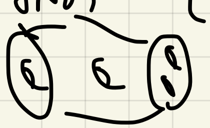
$(N^{d-1}_{-}, g_{-}) \longrightarrow (N^{d-1}_{+}, g_{+})$



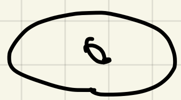
$I : \mathrm{Bord}^{S}_{(d-1,d)} \to \mathrm{Vect}_{\mathbb{C}}$ gives ...

- (N^{d-1}, g_N)  $\longmapsto$ $I(N^{d-1}, g_N)$: $\mathbb{C}$ - vect. sp.

  $= \emptyset \longmapsto I(\emptyset) = \mathbb{C}$ (with trivialization)

- (N_0, g_{N_0}) (N_1, g_{N_1})  (M, g_M) $\longmapsto$ $I(N_0, g_{N_0}) \xrightarrow{I(M^d, g_M)} I(N_1, g_{N_1})$: linear map.

- For (M^d, g_M) : **closed d-dim** S-mfd,

  $\emptyset$  $\emptyset$ (M^d, g_M) $\longmapsto$ $I(M^d, g_M) : \mathbb{C} \to \mathbb{C}$

$\rightsquigarrow$ $Z_I(M^d, g_M) := I(M^d, g_M) \in \mathbb{C}$

Partition function for (M^d, g_M)

(分配関数)

Examples of QFT's.

① Holonomy theory (1) (invertible, non-topological)

... (0, 1)-dim. $\mathcal{S}$: orientation + U(1)-bundle + connection

$\Leftrightarrow$ herm. line bundle w/ connection (L, ∇)

$$\mathrm{Hol} : \mathrm{Bord}_{(0,1)}^{SO \times U(1)_\nabla} \to \mathrm{Line}_{\mathbb{C}} \quad \text{given by :}$$

$\mathrm{Line}_{\mathbb{C}}$: cat. of 1-dim $\mathbb{C}$-vec. sp. $\subset \mathrm{Vect}_{\mathbb{C}}$

· $$\left(\begin{array}{c} L_N \\ \downarrow \\ \bullet\, N \\ +/- \end{array} \right) \longmapsto L_N$$

· $$\left(\begin{array}{c} (L_M, \nabla_M) \\ \downarrow \\ \partial_- M \bullet \xrightarrow{\quad M \quad} \bullet\, \partial_+ M \end{array} \right) \longmapsto \mathrm{Hol}_{\nabla_M} : L_{\partial_- M} \to L_{\partial_+ M}$$

Hol_{∇_M} : parallel transport by ∇_M

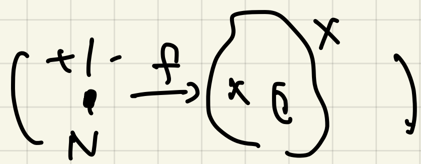
$$\to Z_{\mathrm{Hol}} \left(\begin{array}{c} (L, \nabla) \\ \downarrow \\ S^1 \end{array} \right) = \underset{\in U(1)}{\mathrm{Hol}(\nabla)} \ : \text{Holonomy}$$

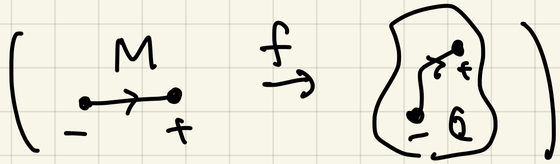
② Holonomy theory (2) (invertible, non-topological)

Fix (L, ∇) ← Line bundle + connection
$\downarrow \mathbb{C}$
X ← a manifold

→ $(0,1)$-dim QFT S : orientation + map to X

$$\mathrm{Hol}_{(X,\nabla)} : \mathrm{Bord}^{SO}_{(0,1)}(X) \to \mathrm{Line}_{\mathbb{C}}$$

-  $\mapsto L_{f(N)}$

-  $\mapsto \mathrm{Hol}_\nabla : L_{f(\partial_- M)} \to L_{f(\partial_+ M)}$

$\rightsquigarrow Z_{\mathrm{Hol}}($  $) = \mathrm{Hol}_f(\nabla) \in U(1)$ Holonomy along a loop.

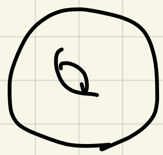
※ ∇_0, ∇_1 : different connections

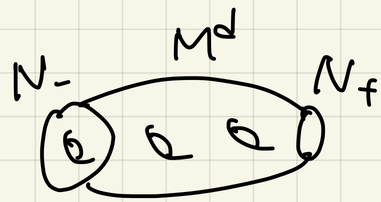
$\rightsquigarrow \mathrm{Hol}_{(X,\nabla_0)} \neq \mathrm{Hol}_{(X,\nabla_1)}$ different theories.

③ Characteristic numbers (invertible, topological)

Fix $\varepsilon \in \mathbb{C}\setminus\{0\}$.

• Euler theory $\mathrm{Eul}_\varepsilon : \mathrm{Bord}_{(d-1,d)} \to \mathrm{Line}_\mathbb{C}$

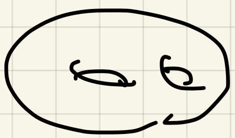
( N^{d-1}) $\mapsto$ $\underline{\mathbb{C}}$ (trivial)

(

) $\mapsto$ $\varepsilon^{\chi(M)} \cdot : \underline{\mathbb{C}} \to \underline{\mathbb{C}}$

& Euler number.

functor, because: ($\chi(M_1 \cup_N M_2) = \chi(M_1) + \chi(M_2)$)

$\rightsquigarrow Z($  $M^d) = \varepsilon^{\chi(M)}$.

Same for other characteristic numbers,

• Signature theory $\mathrm{Sign}_\varepsilon : \mathrm{Bord}^{SO}_{(d-1,d)} \to \mathrm{Line}_\mathbb{C}$

- ...

④ Finite gauge theory (topological)

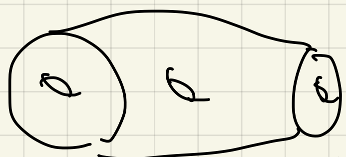
Fix H : a finite group.

$\mathrm{Bun}_H(X) := \{\text{principal } H\text{-bundles over } X\} / \text{isom.}$

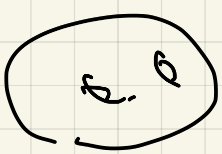
$\to$ $(d-1, d)$-dim QFT. (no S)

$$\mathcal{G}_H : \mathrm{Bord}_{(d-1,d)} \to \mathrm{Vec}_{\mathbb{C}}$$

- ( N^{d-1}) $\mapsto \mathcal{G}_H(N) = \mathrm{Map}(\mathrm{Bun}_H(N), \mathbb{C})$

- ( N_-^{d-1} M^d N_+^{d-1}) $\rightsquigarrow$ $\left(\begin{tikzcd} & \mathrm{Bun}_H(M) \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \mathrm{Bun}_H(N_-) & & \mathrm{Bun}_H(N_+) \end{tikzcd} \right)$

$\mapsto t_* \circ s^* : \mathcal{G}_H(N_-) \to \mathcal{G}_H(N_+)$

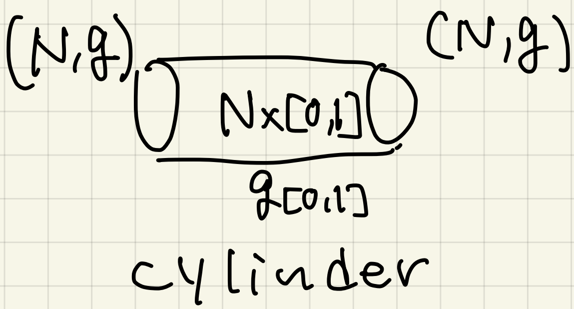
$Z($  M^d $)$ $= \sum_{P \in \mathrm{Bun}_H(M)} \frac{1}{\# \mathrm{Aut}(P)}$

★ **Topological** v.s. **Non-Topological**

(TQFT)

↑ { G-structure
H-bundle, map to X

↑ { G-str / H-bundle with connections,
metrics,
map to X …



topological

$\mathrm{Id} : I(N,g) \to I(N,g)$

non-topological

$I(N,g) \xrightarrow{\neq \mathrm{Id}} I(N,g)$

"time evolution"

- ∃ Aixiom for TQFT : Atiyah-Segal's axiom.

⇝ Mathematical approach is easier.

(but very nontrivial & interesting!)

- No (agreed) axiom for non-topological QFT.

Classifications for QFT's ?

$\Leftarrow$ up to "deformation equivalence"
(変形同値)

QFT's I_0 & I_1 are **deformation equivalent** $I_0 \sim I_1$
"$\underset{\text{def}}{\Longleftrightarrow}$" $\exists \{I_t\}_{t\in[0,1]}$ conti. path of QFT's from I_0 to I_1

<u>Ex</u> ② Hol. theory (2). $\mathrm{Hol}_{(L,\nabla)}$ with $\begin{array}{c}(L,\nabla)\\ \downarrow\\ X\end{array}$ target

two connections ∇_0, ∇_1 on $L \to X$

$\rightsquigarrow$ $\mathrm{Hol}_{(L,\nabla_0)} \sim \mathrm{Hol}_{(L,\nabla_1)}$

③ Euler theory Eul_ε, $\varepsilon \in \mathbb{C}\setminus\{0\}$.

$\varepsilon_0, \varepsilon_1 \in \mathbb{C}\setminus\{0\}$ $\rightsquigarrow$ $\mathrm{Eul}_{\varepsilon_0} \sim \mathrm{Eul}_{\varepsilon_1}$.

General "Conjecture" (Kitaev, Freed-Hopkins, ⋯)

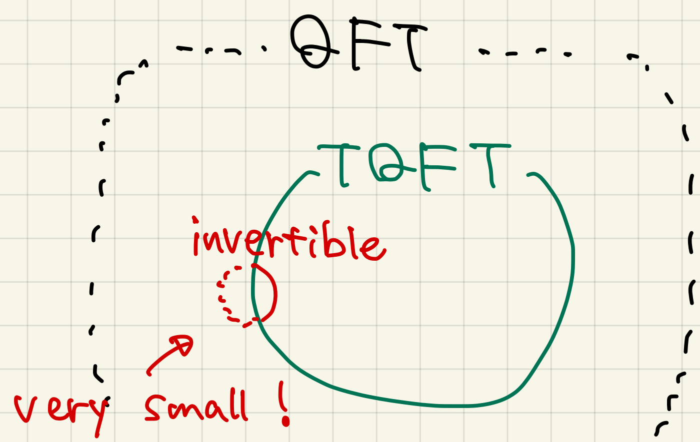
"Invertible QFT's should be classified by Generalized Cohomology theories."

※ Invertible $\subset$ non-invertible

$\underset{\text{def}}{\Leftrightarrow}$ the domain lies in $\mathrm{Line}_{\mathbb{C}}$: cat of 1-dim $\mathbb{C}$-vec. sp.

$$\mathrm{Bord}^{S}_{(d-1,d)} \xrightarrow{I} \mathrm{Line}_{\mathbb{C}} \subset \mathrm{Vect}_{\mathbb{C}}$$

(※ Appear as "low energy effective theories" in physics)



§3. Our Results

We explain this result :

一般コホモロジーは抽象的 … そこで

物理的な（QFTと直接結びつく）定式化を与えよう！

- $I\Omega^G$ (Anderson dual to G-bordism) という一般コホモロジーに対して，invertible QFTを抽象化したモデルを与えた．

(米倉-Y, 2021)

$\underline{I\Omega^G}$: the Anderson dual to G-bordism.

a generalized cohomology satisfying

$$0 \to \mathrm{Ext}(\Omega^G_d(X), \mathbb{Z}) \to (I\Omega^G)^{d+1}(X) \to \mathrm{Hom}(\Omega^G_{d+1}(X), \mathbb{Z}) \to 0$$

(exact)

$$\Omega^G_d(X) := \left\{ (M^d, g, f) \;\middle|\; \begin{array}{l} M: \text{d-dim. closed manifold} \\ g: \text{stable G-str. on } TM \\ f: M \to X \text{ conti} \end{array} \right\} \Big/ \underset{\text{bordism}}{\sim}$$

☆ Starting point : $\boxed{\text{Conjecture}}$ (Freed - Hopkins '16)

$$\left\{ \begin{array}{l} \text{possibly non-topological} \\ \text{invertible d-dim QFT} \\ \text{on } G_\nabla\text{-manifolds} \end{array} \right\} \Big/ \underset{\text{deformation equivalence}}{\sim} \overset{1:1}{\longleftrightarrow} \underline{(I\Omega^G)}^{d+1}(pt)$$

a generalized cohomology theory called "Anderson dual to G-bordism"

One difficulty of $\boxed{\text{Conj}}$: the def. of $I\Omega^G$ is abstract...

⇝ We give a new approach to $\boxed{\text{Conj}}$ by overcoming it.

☆ <u>Main Result</u> : Construction of a new model of $(I\Omega^G)^*$

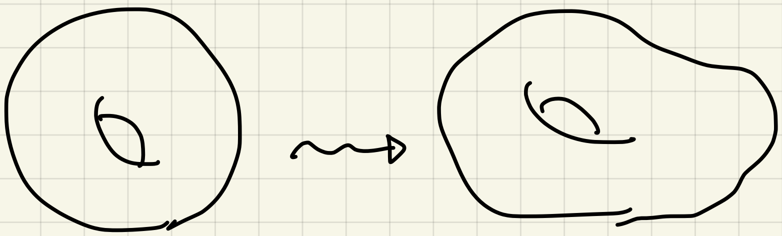
made by some "Empirical Facts" (経験則) on QFTs

※ related to differential cohomology theories.

An "Empirical Fact" on invertible QFT:

($(d-1,d)$-dim)

"Partition functions varies locally".

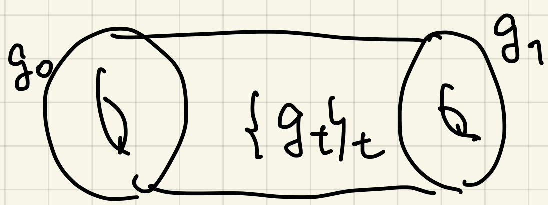


$(M^d, g_0) \rightsquigarrow (M^d, g_t)$ smooth family $\{g_t\}_t$

$$\Rightarrow \frac{d}{dt} \log Z_I(M, g_t) = 2\pi i \int_M \beta(M, \{g_t\}_t)$$

$\exists\, \beta(M, \{g_t\}_t) \in \Lambda^d(M)$ differential form locally constructed from $\{g_t\}$.

$\Downarrow$ In other words...



M $\quad I\times M$

$\Uparrow$

$(d+1)$-dim

$$\to \omega_I(I\times M, \{g_t\}_t) = dt \wedge \beta(M, \{g_t\}_t)$$

$$\rightsquigarrow \frac{Z(M, g_1)}{Z(M, g_0)} = \exp\left(2\pi i \int_{I\times M} \omega_I(I\times M, \{g_t\})\right)$$

More generally, it is expected that :

(d-1, d)-dim invertible QFT I $\underset{\text{(up to deformation)}}{\overset{\text{associates}}{\rightsquigarrow}}$ (ω_I, Z_I)

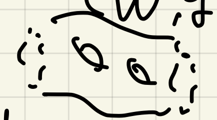
- Z_I : (M^d, g_M) d-dim clo. S-mfd

(M^d, g_M)

$\longmapsto$ Partition function $Z_I(M^d, g_M) \in U(1)$

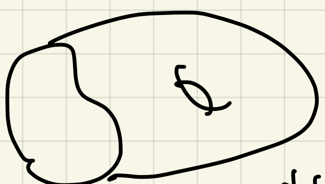
- ω_I : (W^{d+1}, g_W) (d+1)-dim S-mfd

(W^{d+1}, g_W)



$\longmapsto$ "local" procedure $\omega_I(W^{d+1}, g_W)$ $\in$ $\Lambda^{d+1}(W)$ differential form.

With compatibility : Given



(M^d, g_M) (W^{d+1}, g_W)

$\parallel$

$\partial(W, g_W)$

, We have

$$Z_I(M, g_M) = \exp\left(2\pi i \int_W \omega_I(W, g_W)\right)$$

Result Set S = G_∇ - structure + map to X

Thm (Rough version. 米倉 - Y. 2021)

Based on this "fact," define a group

$$(I\Omega^G_{dR})^{d+1}(X) := \text{“}\left\{ (\omega_I, Z_I) \,\middle|\, \begin{array}{l} I:\ d\text{-dim invertible QFT} \\ \text{on } G_\nabla\text{-manifold with map to } X \end{array} \right\} \Big/ \underset{\text{def}}{\sim} \text{”}$$

→ this is actually a model for $I\Omega^G$.

$\widehat{(I\Omega^G_{dR})}^{d+1}(X)$: "differential cohomology theory"

The Physical meaning of

$$0 \to \mathrm{Ext}(\Omega^G_d(X), \mathbb{Z}) \to (I\Omega^G)^{d+1}(X) \to \mathrm{Hom}(\Omega^G_{d+1}(X), \mathbb{Z}) \to 0 \quad : \text{(exact)}$$

$\mathrm{Ext}(\Omega^G_d(X), \mathbb{Z}) = \mathrm{Hom}(\Omega^G_d(X), \mathbb{R}/\mathbb{Z}) \big/ \mathrm{Hom}(\Omega^G_d(X), \mathbb{R})$

$$[\omega_I, Z_I] \mapsto \left\{ [W, g_W] \mapsto \int_W \omega_I(g_W) \right\}, \quad \int_W \omega_I(g_W) \in \mathbb{Z}$$

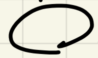
$$Z_{top} : \Omega^G_d(X) \to \mathbb{R}/\mathbb{Z} \ \mapsto\ [0, Z_{top}]$$

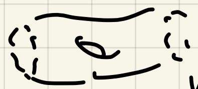
↑ a class of an invertible TQFT

Applications

$$
\begin{array}{ccc}
\textbf{Phys} & & \textbf{Math} \\[1ex]
\left\{ \begin{array}{l} \text{invertible } (d-1,d)\text{-QFT} \\ \text{on } G_\sigma\text{-mfds.} \\ \quad \text{with map to } X \end{array} \right\} \Big/ \sim_{\text{deformation}} & \xleftrightarrow{\text{Empirical fact}} & \left(I\Omega^G_{dR}\right)^{d+1}(X) \\
 & & \Vert \ \boxed{\text{Thm}} \\
 & & \left(I\Omega^G\right)^{d+1}(X) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
[I] & \longmapsto & [(\omega_I, z_I)] \\
\text{known QFT} & \rightsquigarrow & \text{Mathematical element !}
\end{array}
$$

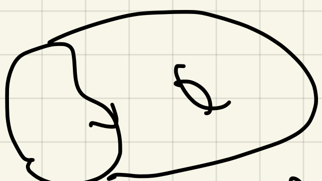
① Holonomy theory (1) : $\quad \mathrm{Hol} : \mathrm{Bord}_{\langle 0,1 \rangle}^{SO \times U(1) \nabla} \to \mathrm{Line}_{\mathbb{C}}$

- $Z_{\mathrm{Hol}}\left(\begin{array}{c} (L, \nabla) \\ \downarrow \\ S^1 \end{array} \right) = \mathrm{Hol}(\nabla)$ 

- $\omega_{\mathrm{Hol}}\left(\begin{array}{c} (L, \nabla) \\ \downarrow \\ W^2 \end{array} \right) := \frac{1}{2\pi} \mathrm{curv}(\nabla) = c_1(\nabla) \in \Lambda^2(W)$ 

$\uparrow$ curvature of ∇

Compatibility :



$(M^1, \nabla_M) \quad (W^2, \nabla_W)$

$\parallel$

$(\partial W, \nabla_W|_\partial)$

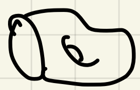
$\rightsquigarrow \quad \mathrm{Hol}(\nabla_M) = \exp\left(\int_W \mathrm{curv}(\nabla_W) \right)$

$$\rightsquigarrow \quad (c_1, \mathrm{Hol}) \in \left(I\Omega_{dR}^{SO \times U(1)} \right)^2 (pt)$$

② Holonomy theory (2) : $\mathrm{Bord}^{SO}_{\langle 0,1 \rangle}(X) \to \mathrm{Line}_{\mathbb{C}}$

$$\begin{array}{c} (L, \nabla_X) \\ \downarrow \\ X \end{array}$$

- $Z_{\mathrm{Hol}(X,\nabla)}\Big($  $\Big) = \mathrm{Hol}_{\nabla_X} \in U(1)$

- $W_{\mathrm{Hol}(X,\nabla)}\Big($  $\xrightarrow[f]{} X\Big) = f^* c_1(\nabla_X) \in \Lambda^2_{clo}(W)$

$$\rightsquigarrow \quad (c_1(\nabla_X), \mathrm{Hol}_{\nabla_X}) \in (\widehat{I\Omega^{SO}_{dR}})^2(X)$$

↑ "differential cohomology"

∇_X, ∇_X' : two connections on $L \to X$

$$\rightsquigarrow \quad [c_1(\nabla_X), \mathrm{Hol}_{\nabla_X}] = [c_1(\nabla_X'), \mathrm{Hol}_{\nabla_X'}] \in (I\Omega^{SO}_{dR})^2(X)$$

↑

Reflects the deformation equivalence "$\mathrm{Hol}_{\nabla_X} \overset{\mathrm{def}}{\sim} \mathrm{Hol}_{\nabla_X'}$"

③ classical Chern-Simons theory $CS: \mathrm{Bord}^{SO\times H_\nabla}_{(2,3)} \to \mathrm{Line}_{\mathbb{C}}$

Fix H: a compact Lie group.

$$Z_{CS}\left(\begin{array}{c} P, \nabla_M \\ \downarrow \\ M^3 \end{array}\right) = \text{Chern-Simons invariant for } \nabla_W$$

$$\omega_{CS}\left(\begin{array}{c} P, \nabla_W \\ \downarrow \\ W^4 \end{array}\right) = ch_2(\nabla_W) \in \Lambda^4_{clo}(W) \quad \text{2nd Chern character form}$$

$$\to (ch_2, Z_{CS}) \in (I\Omega^{SO\times H}_{dR})^4(pt)$$

※ If $P = M \times H$, $\nabla_A = d + A$

$$Z_{CS}(M, P, \nabla_A) = \exp\left(2\pi i \int_M \underbrace{Tr\left(dA\wedge A + \tfrac{2}{3} A\wedge A\wedge A\right)}_{= CS(\nabla_A) \in \Lambda^3(M)}\right)$$

$$ch_2(W, P, \nabla_W) = Tr\left((dA\wedge A + A\wedge A)^2\right)$$

④ Massive Free Fermions , $(d = 8k+3)$

$S = \mathrm{Spin}\nabla$

$\rightsquigarrow \left(\frac{1}{2}\hat{A},\ Z_\eta\right)$, where

eta inv for D

$$\bar{\eta}(D) = \frac{\eta(D) + \dim\ker(D)}{2}$$

- $Z_\eta(M^{8k+3}, s, \nabla) = \exp\left(\pi i\, \bar{\eta}(\not{D}_{M,\nabla})\right)$

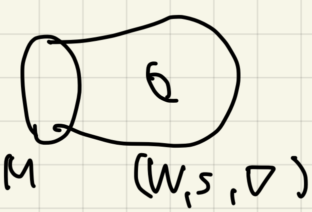
spin Dirac operator on (M, s, ∇)

- $\frac{1}{2}\hat{A}(W^{8k+4}, s, \nabla) = \frac{1}{2}\cdot\hat{A}(\nabla)\big|_{8k+4} \in \Omega^{8k+4}(W)$

☆ Compatibility : Atiyah-Patodi-Singer index thm

$$\mathrm{Ind}_{APS}(\not{D}_{W,\nabla}) = \int_W \hat{A}(W, s, \nabla) - \bar{\eta}(\not{D}_{M,\nabla})$$

$\mathrm{Ind}_{APS}(\not{D}_{W,\nabla}) \in 2\mathbb{Z}$ because $\dim W = 8k+4$



$$\rightsquigarrow \left(\tfrac{1}{2}\hat{A}\big|_{8k+4},\ \exp(\pi i\, \bar{\eta})\right) \in \left(I\Omega^{\mathrm{Spin}}_{dR}\right)^{8k+4}(pt)$$