THE ALEXANDER POLYNOMIAL OF A TRIVALENT SPATIAL GRAPH AND ITS MOY-TYPE RELATIONS

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Unlike the Alexander polynomial of a knot, there is no standard definition for the Alexander polynomial of a spatial graph. In this talk, we study a version of it. Let G be an oriented trivalent graph without source or sink embedded in S^3, and c a positive balanced coloring of G. We define a topological invariant $\Delta_{(G,c)}(t)$ for G and study five interpretations for it. The contents of the talk come from our preprints [3, 2].

Here are five interpretations.

(i) The balanced coloring c of G naturally defines a homomorphism

$$\begin{array}{rcl} \phi_c : \pi_1(S^3\backslash G, x_0) & \to & H_1(S^3\backslash G;\mathbb{Z}) \to \mathbb{Z}\langle t\rangle \\ \text{oriented meridian of } e & \mapsto & t^{c(e)}, \end{array}$$

where $\mathbb{Z}\langle t\rangle$ is the abelian group generated by t. Let $X = S^3\backslash G$. Then $\ker(\phi_c)$ corresponds to a regular covering space of X, which we call $p : \widetilde{X} \to X$. Consider the $\mathbb{Z}[t^{-1}, t]$-module $H_1(\widetilde{X}, p^{-1}(\partial_{\text{in}}(X)))$, where $\partial_{\text{in}}(X) := \bigcup_{v\in V} \partial_{\text{in}}(v) \subset \partial(X)$ and $\partial_{\text{in}}(v)$ is a subsurface around vertex v bounded by meridians of the edges pointing toward v and the "meridian" circle around v. The polynomial $\Delta_{(G,c)}(t)$ is defined to be the 0-th characteristic polynomial of a presentation matrix of the module. It is a topological invariant of G modulo $\pm\mathbb{Z}\langle t^{1/2}\rangle$.

(ii) We studied the Heegaard Floer homology for a balanced bipartite graph with a balanced orientation ([1]). If we deform a trivalent graph into a bipartite graph by inserting an edge on each vertex, the Euler characteristic of the Heegaard Floer homology for the resulting bipartite graph is $\Delta_{(G,c)}(t)$.

(iii) Kauffman ([4]) studied a state sum model for the Alexander polynomial of a knot, where a state is a one-one correspondence between the set of crossings and the set of unmarked regions on a knot diagram. We extend his idea and provide a state sum model for $\Delta_{(G,c)}(t)$, where new types of crossings and regions around a vertex are introduced.

(iv) $\Delta_{(G,c)}(t)$ satisfies a series of relations, which we call MOY-type relations. These relations are inspired by Murakami-Ohtsuki-Yamada's relations in [5], where they provided a graphical definition for the $U_q(\mathfrak{sl}_n)$-polynomial invariants of a link for all $n \geq 2$. We show that these relations also provide a graphical definition for the Alexander polynomial of a link, thus extending MOY's graphical calculus to the case $n = 0$. In addition, these relations characterize $\Delta_{(G,c)}(t)$ for a framed trivalent graph G.

(v) Viro [6] defined a functor from the category of colored framed trivalent graph to the category of finite dimensional modules over the q-deformed universal enveloping algebra $U_q(gl(1|1))$, and constructed the $gl(1|1)$-Alexander polynomial of a graph from the functor. We show that Viro's Alexander polynomial satisfies an adapted version of MOY-type relations, and thus coincides with $\Delta_{(G,c)}$.

References

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