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Representation theory of Lie superalgebras
(スーパー Lie 環の表現論)

What are the possible solutions of the following equation?
(次の方程式の解は何でしょうか?)

$$XY + YX = 1$$

If X and Y are real or complex numbers, then X and Y commute.
(もし X と Y が実数か複素数なら X と Y は可換です.)

The solutions are: $X \neq 0$ and $Y = \frac{1}{2X}$.
(このとき解は $X \neq 0$, $Y = 1/2X$ です.)

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be the 2×2 identity matrix.
(I を 2 行 2 列の単位行列としましょう.)

If X and Y are 2×2-matrices, then we have solutions of the equation
(X と Y が 2 行 2 列の行列なら, 方程式 $XY + YX = I$ の)

$XY + YX = I$, where X and Y do not commute.
(非可換な解を得ることができます.)

For example:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

since

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Let M_r be the set of $r \times r$-matrices.
(M_r を $r \times r$ 行列全体のなす集合としましょう.)

If we want to answer the question:
(そして次の問題を解きたいと思ったとしましょう.)

"What are the matrix solutions of the equation: $XY + YX = I$?"
(方程式 $XY + YX = I$ を満たす行列は何か?)

we can ask: "What are the maps $f : \{X, Y, I\} \to M_r$ such that
(次のように問うこともできます: 写像 $f : \{X, Y, I\} \to M_r$ であって)

$f(X), f(Y), f(I)$ satisfy the following system of equations?"
($f(X), f(Y), f(I)$ が次を満たすものは何か?)

$$\begin{aligned} f(X)f(Y) + f(Y)f(X) &= f(I) \\ f(I)f(X) - f(X)f(I) &= f(0) = 0 \\ f(I)f(Y) - f(Y)f(I) &= f(0) = 0 \end{aligned}$$

A better way to answer this question is: Give the set $\{X, Y, I\}$ the
(この問題に答えるもっと良いやり方は次の通りです:)

structure of an algebra "L" where multiplication is defined by these
(集合 $\{X, Y, I\}$ に積構造が上の式で定まる代数 "L" の構造を与え,)

equations, and give the set M_r the structure of an algebra "$\mathfrak{gl}(r|s)$",
(さらに集合 M_r に代数 "$\mathfrak{gl}(r|s)$" の構造を与え,)

and then ask:
(次の問いに答えればいいのです.)

"What are the algebra maps $f : L \to \mathfrak{gl}(r|s)$?"
(代数の準同型写像 $f : L \to \mathfrak{gl}(r|s)$?" は何か?)

We will define these algebras and try to make this more clear.
(これからこれらの代数を定義し, これらのことをもっと詳しく説明します.)

An **algebra** A is a vector space with multiplication * satisfying:

(代数 A とはベクトル空間であって次を満たす積 $*$ が定義されているものをいいます:)

$$\begin{aligned}(x+y)*z &= x*z+y*z \qquad\qquad x,y,z\in A\\ x*(y+z) &= x*y+x*z\\ (ax)*(by) &= (ab)(x*y) \qquad\qquad a,b\in\mathbb{R}.\end{aligned}$$

For example: M_r is an (associative) algebra where the

(例えば, M_r は通常の行列の掛け算により (結合) 代数になります.)

multiplication is given by the usual product of matrices.

A **superalgebra** is an algebra $A = A_0 \oplus A_1$, where the multiplication

(スーパー代数とは代数 $A = A_0 \oplus A_1$ であって, 積が次を満たすもの)

satisfies

(をいいます:)

- if $x, y \in A_0$, then $x * y \in A_0$;
- if $x \in A_0$ and $y \in A_1$, then $x * y \in A_1$;
- if $x, y \in A_1$, then $x * y \in A_0$.

Elements of A_0 are called **even** and elements of A_1 are called **odd**.

(A_0 の元は even(偶) と呼ばれ, A_1 の元は odd(奇) と呼ばれます.)

For example: $M_{r+s} = A_0 \oplus A_1$ is a superalgebra where

(M_{r+s} を $M_{r+s} = A_0 \oplus A_1$ と分解することによりスーパー代数だと思えます. ここで,)

$$A_0 = \{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in M_r,\ B \in M_s\}$$

$$A_1 = \{\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \mid C \text{ is } (r\times s)\text{-matrix},\ D \text{ is } (s\times r)\text{-matrix}\}$$

A **Lie superalgebra** is a superalgebra $A = A_0 \oplus A_1$ where the
(スーパー Lie 環とはスーパー代数 $A = A_0 \oplus A_1$ であって,)
multiplication, satisfies two additional axioms:
(積がさらに次の公理を満たすものを言います:)

(1) $[X, Y] = -(-1)^{ij}[Y, X]$, where $X \in A_i$, $Y \in A_j$;

(2) $[X, [Y, Z]] = [[X, Y], Z] + (-1)^{ij}[Y, [X, Z]]$.

For example: $\mathfrak{gl}(r|s) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$

Let us define a new multiplication on M_{r+s}.
(M_{r+s} に次で新しい積を導入しましょう.)

$M_{r+s} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a superalgebra with the matrix multiplication.
($M_{r+s} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ と分解し, $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$ なら)

If $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$, define $[X, Y] = XY - (-1)^{ij}YX$.
($[X, Y] = XY - (-1)^{ij}YX$ と定義するのです.)

With this multiplication M_{r+s} is the Lie superalgebra $\mathfrak{gl}(r|s)$.
(この積により M_{r+s} はスーパー Lie 環になりますがこれを $\mathfrak{gl}(r|s)$ と書きます.)

For example: $\mathfrak{gl}(1|1)$

$$\mathfrak{g}_0 = \mathrm{span}\{A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$$
$$\mathfrak{g}_1 = \mathrm{span}\{X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$$

$$[X, Y] = XY + YX = A + B$$
$$[A, X] = AX - XA = X \qquad [B, X] = BX - XB = -X$$
$$[A, Y] = AY - YA = -Y \qquad [B, Y] = BY - YB = Y$$

A **representation** of a Lie superalgebra $L = L_0 \oplus L_1$ is a linear map
(スーパー Lie 環 $L = L_0 \oplus L_1$ の表現とは線型写像)

$$\rho : L \to \mathfrak{gl}(r|s)$$

which preserves the multiplication: $\rho([X, Y]) = [\rho(X), \rho(Y)]$, and
(であって積構造を保つものを言います:)

such that $\rho(L_0) \subset \mathfrak{g}_0$ and $\rho(L_1) \subset \mathfrak{g}_1$.

For example: Let $L = L_0 \oplus L_1$ be the three dimensional Lie

superalgebra with vector space basis $\{X, Y, H\}$ such that:

$$L_0 = \mathrm{span}\{H\}, \qquad L_1 = \mathrm{span}\{X, Y\},$$

and multiplication relations:

$$[X, Y] = H, \qquad [H, X] = 0, \qquad [H, Y] = 0.$$

There is a representation $\rho : L \to \mathfrak{gl}(1|1)$ given by:

$$\rho(X) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \rho(Y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \rho(H) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is the same as our first system of equations and solution.

Using this language we can systematically study all representations
(この言語を使うことにより, 我々はスーパー Lie 環 L の全ての表現の)

of a Lie superalgebra L (“study solutions of the system of equations”).
(組織的な研究をすることができます (“方程式系の解の研究”))

Recall that a $(r \times r)$-matrix defines a linear endomorphism $f : V \to V$
($r \times r$ 行列が r 次元ベクトル空間 V 上の線型写像)

of an r-dimensional vector space V.
($f : V \to V$ を定めることを思い出しましょう.)

So we are also studying endomorphisms of an $(r + s)$-dimensional
(従って我々は定義方程式を解く $r + s$ 次元ベクトル空間上の)

vector space which solve the defining equations.
(線型写像を研究していることになります.)

Sophisticated techniques have been developed to study the
(スーパー Lie 環の表現を研究する洗練されたテクニックが開発され,)

representations of a Lie superalgebra, which have led to lots of
(たくさんの興味深く, 数学的に深い内容の結果が得られました.)

interesting and deep mathematics.

Lie superalgebras first appeared in mathematics in the context of
(スーパー Lie 環が初めて数学に登場したのは, 1950 年代の終りの)

“deformation theory” in the work of A. Frölicher and A. Nijenhuis
(A. Frölicher and A. Nijenhuis による変型理論の中です.)

in the late 1950s.

In 1977, Victor Kac completely classified all finite dimensional simple
(1977 年に Victor Kac は複素数体上の全ての有限次元単純スーパー)

Lie superalgebras over the complex numbers. (A Lie superalgebra L
(Lie 環を分類しました. (スーパー Lie 環 L は自分自身と零しか)

is called simple if the only "ideals" in L are $\{0\}$ and L.)
(イデアルを持たないとき単純であると呼ばれます.))

Kac also showed that the finite dimensional "irreducible"
(Kac はまた単純スーパー Lie 環の有限次元既約表現は皆,)

representations of simple Lie superalgebras are "highest weight
(最高ウエイト λ がある条件を満たす最高ウエイト表現であること)

representations" given by certain conditions on the "weight" λ.
(を示しました.)

The "character formula" is an extremely important tool for studying
(指標公式はスーパー Lie 環を研究する上で極めて重要な道具です.)

the representations of a Lie superalgebra.

In 1978, Victor Kac found the character formula for finite
(1978 年に Victor Kac は "generic" な最高ウエイトを持つ)

dimensional representations with "generic" highest weight.
(有限次元既約表現の指標公式を発見しました.)

In 1996, Vera Serganova found the character formula for finite
(1996 年に Vera Serganova は任意の有限次元表現の指標公式を)

dimensional representations with arbitrary weights.
(発見しました.)

There are two types of finite dimensional simple Lie superalgebras:
(有限次元単純スーパー Lie 環には二つのタイプがあります.)

"classical type" and "Cartan type".
(古典型とカルタン型です.)

Kac-Moody superalgebras are a generalization of the classical type.
(Kac-Moody スーパー Lie 環は古典型の一般化です.)

Kac-Moody superalgebras can be infinite dimensional.
(Kac-Moody スーパー Lie 環は無限次元になり得ます.)

A Kac-Moody superalgebra $\mathfrak{g}(A)$ is a Lie superalgebra defined by
(Kac-Moody スーパー Lie 環 $\mathfrak{g}(A)$ は生成元と関係式で定義された)

generators and relations, where A is a matrix which encodes
(スーパー Lie 環です. ここで, A は定義関係式の)

information about the defining relations.
(情報を与える行列です.)

A is called the **Cartan matrix** of $\mathfrak{g}(A)$.
(A は $\mathfrak{g}(A)$ の Cartan 行列と呼ばれます.)

For example: Let $A = \begin{pmatrix} 2 \end{pmatrix}$. The Kac-Moody superalgebra

$\mathfrak{g}(A)$ is generated by even elements X, Y, H, which satisfy the

following relations:

$$[X, Y] = H$$
$$[H, X] = 2X$$
$$[H, Y] = -2Y.$$

Since a Kac-Moody superalgebra can be infinite dimensional, it is
(Kac-Moody スーパー代数は無限次元になり得ますので,)

useful to know: When is the Kac-Moody superalgebra “not too big”?
(いつ「大きくなりすぎない」かを知ることは有益です.)

The condition of “finite growth” answers this question.
(「有限成長性」の条件がこの問いに答えます.)

In 1978, Victor Kac classified Kac-Moody superalgebras with finite
(1978 年に Victor Kac は Cartan 行列の対角成分に零が含まれない)

growth, with the additional assumption that the Cartan matrix has
(という条件の下で, 有限成長な Kac-Moody 代数の)

no zeros on the main diagonal.
(分類を与えました.)

In 1986, van de Leur classified Kac-Moody superalgebras with finite
(1986 年に van de Leur は, Cartan 行列が対称化可能である,)

growth, with the additional assumption that the Cartan matrix is
(という条件の下で, 有限成長な Kac-Moody 代数を分類しました.)

“symmetrizable”.

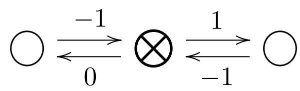
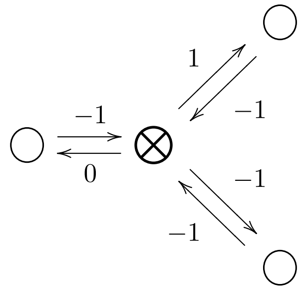
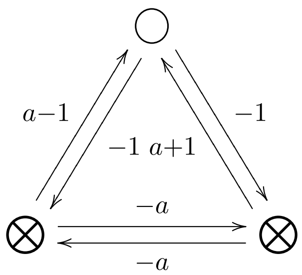
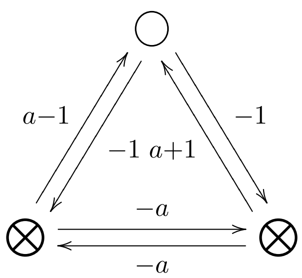
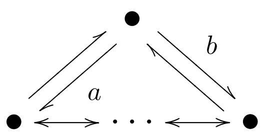
A matrix A is called **symmetrizable** if there exists an invertible
(行列 A は $A = DB$ を満たす可逆な行列 D と対称行列 D が)

diagonal matrix D and a symmetric matrix B such that $A = DB$.
(存在するときに対称化可能であるといいます.)

In 2007, Crystal Hoyt and Vera Serganova completely classified
(2007 年に Crystal Hoyt と Vera Serganova は有限成長な)

Kac-Moody superalgebras with finite growth.
(Kac-Moody スーパー代数の完全な分類を与えました.)

Non-symmetrizable Kac-Moody Superalgebras with Finite Growth

Algebra	**Dynkin diagrams**
$D(2,1,0)$	
$\widehat{D}(2,1,0)$	
$S(1,2,0)$	 $a \in \mathbb{Z}$
$S(1,2,a)$	 $a \notin \mathbb{Z}$
$q(n)^{(2)}$	 There are n $\bullet$. Each $\bullet$ is either $\bigcirc$ or $\bigotimes$. An odd number of them are $\bigotimes$. If $\bullet$ is $\bigcirc$, then $a = b = -1$. If $\bullet$ is $\bigotimes$, then $\frac{a}{b} = -1$.

A “Weyl character formula” for the irreducible representations of an
(無限次元 Kac-Moody スーパー代数の Weyl 指標公式は)

infinite dimensional Kac-Moody superalgebra is still not proven.
(未だに証明されていません.)

Recently, Vera Serganova has proven the Weyl character formula for
(最近, Vera Seranova は次の無限次元 Kac-Moody スーパー代数)

the following infinite dimensional Kac-Moody superalgebras:
(について Weyk 指標公式を証明しました.)

$\mathfrak{sl}(1,n)^{(1)}$ and $\mathfrak{osp}(2,2n)^{(1)}$.

There is still no proven character formula for the remaining
(他の Kac-Moody スーパー代数の指標公式は解かれていません.)

Kac-Moody superalgebras.

This is a very important open question in the representation theory
(これは極めて重要なオープンな問題です.)

of Lie superalgebras.