Some property of harmonic transformed one dimensional generalized diffusion processes

TOMOKO TAKEMURA
Nara Women's University
joint work with Matsuyo Tomisaki

2009. 6.29

1. harmonic transform

X : one dimensional diffusion process on $I = (l_1, l_2)$

s : scale function

m : speed measure function

k : killing measure function

$\mathcal{G}_{s,m,k}$: generator of X

We consider

$$\mathcal{H}^*_{s,m,k,\beta} = \{ \; h \mid \mathcal{G}_{s,m,k} h = \beta h, \; h > 0\}, \quad \beta \geq 0.$$

Remark.

$$\mathcal{H}^*_{s,m,k,\beta} \neq \emptyset$$

$p(t,x,y)$: transition probaility of X w.r.t. m

For $h \in \mathcal{H}^*_{s,m,k,\beta}$, we put

$$p^*(t,x,y) = e^{-\beta t}\frac{p(t,x,y)}{h(x)h(y)}.$$

known result

We define s_h, m_h.

$$s_h(x) = \int_c^x h(x)^{-2}\,ds(x), \quad m_h(x) = \int_c^x h(x)^2\,dm(x), \quad (\ c \in I\ ).$$

Y : one dimensional diffusion process which generator is $\mathcal{G}_{s_h,m_h,0}$,
p^* : the ransition probability of Y w.r.t. m_h.

For $h \in \mathcal{H}^*_{s,m,k,\beta}$,

$$H^*_h : \mathcal{G}_{s,m,k} \mapsto \mathcal{G}_{s_h,m_h,0} \ (H^*\mathcal{G}_{s,m,k} = \mathcal{G}_{s_h,m_h,0}).$$

Miyuki Maeno(2005)

$$\mathcal{H}^o_{s,m,0} = \{h \mid \mathcal{G}_{s,m,0}h \leq 0, \ h > 0\}, \quad (\mathcal{H}^o_{s,m,0} \neq \emptyset)$$

For $h \in \mathcal{H}^o_{s,m,0}$,

$$p^o(t,x,y) = \frac{p(t,x,y)}{h(x)h(y)}.$$

Z : h transform of X

$p^o(t,x,y)$: transition probability of Z w.r.t. dm_h.

$\mathcal{G}_{s_h,m_h,k_h}$: the generator of Z, $(dk_h = -hdD_sh)$

For $h \in \mathcal{H}^o_{s,m,0}$, we put $H^o_h : \mathcal{G}_{s,m,0} \mapsto \mathcal{G}_{s_h,m_h,k_h}$.

Remark.

$\mathcal{H}^*_{s,m,0,0} \subset \mathcal{H}^o_{s,m,0}$.

For $h \in \mathcal{H}^*_{s,m,0,0}, H^o_h \mathcal{G}_{s,m,0} = H^*_h \mathcal{G}_{s,m,0}$.

If $k \neq 0$ or $\beta > 0$, then $\mathcal{H}^o_{s,m,0} \cap \mathcal{H}^*_{s,m,k,\beta} = \emptyset$.

Proposition.

(i) $h \in \mathcal{H}^*_{s,m,k,\beta} \Rightarrow h^{-1} \in \mathcal{H}^o_{s_h,m_h,0}$, $H^o_{h^{-1}} H^*_h \mathcal{G}_{s,m,k} = \mathcal{G}_{s,m,k+\beta m}$
Specially,

$$\beta = 0 \Rightarrow H^o_{h^{-1}} H^*_h \mathcal{G}_{s,m,k} = \mathcal{G}_{s,m,k}$$

(ii) $h \in \mathcal{H}^o_{s,m,0} \Rightarrow h^{-1} \in \mathcal{H}^*_{s_h,m_h,k_h,0}$, $H^*_{h^{-1}} H^o_h \mathcal{G}_{s,m,0} = \mathcal{G}_{s,m,0}$

2. Recurrence and Transience

G : the set of all $\mathcal{G}_{s,m,k}$

Recurrence and Transience

$$G^R = \{\mathcal{G}_{s,m,k} : k = 0,\ s(l_1) = -\infty,\ s(l_2) = \infty\}$$
$$G^T = G \setminus G^R$$

Proposition.

(i) $\mathcal{G}_{s,m,0} \in G^R \Rightarrow \mathcal{H}^o_{s,m,0} = \mathcal{H}^*_{s,m,0,0} = \{c > 0\}$.

(ii) $\mathcal{G}_{s,m,0} \in G^R,\ h \in \mathcal{H}^*_{s,m,0,\beta}\ \ \beta > 0 \Rightarrow H^*_h \mathcal{G}_{s,m,0} \in G^T$

(iii) $\mathcal{G}_{s,m,k} \in G^T,\ h \in \mathcal{H}^*_{s,m,k,\beta},\ \beta \geq 0 \Rightarrow H^*_h \mathcal{G}_{s,m,k} \in G^T$

3. The state of Boundary

Theorem. $h \in \mathcal{H}^*_{s,m,k,\beta}$

	$h(l_i) = 0$	$h(l_i) \in (0, \infty)$	$h(l_i) = \infty$
(s, m, k) -regular	$(s_h, m_h, 0)$ -entrance	$(s_h, m_h, 0)$ -regular	———
(s, m, k) -exit	$(s_h, m_h, 0)$ -entrance	$(s_h, m_h, 0)$ -exit	———
(s, m, k) -entrance	———	$(s_h, m_h, 0)$ -entrance	$(s_h, m_h, 0)$ -regular if $\lvert m_h(l_i)\rvert < \infty$ $(s_h, m_h, 0)$ -exit if $\lvert m_h(l_i)\rvert = \infty$
(s, m, k) -natural	$(s_h, m_h, 0)$ -entrance if $J_{m_h,s_h}(l_i) < \infty$ $(s_h, m_h, 0)$ -natural if $J_{m_h,s_h}(l_i) = \infty$	———	$(s_h, m_h, 0)$ -exit if $J_{s_h,m_h}(l_i) < \infty$ $(s_h, m_h, 0)$ -natural if $J_{s_h,m_h}(l_i) = \infty$

4. Example

$$\mathcal{G}_{s,m,k} = \frac{1}{2}\frac{d^2}{dx^2} - \frac{a^2 - 2^{-2}}{2x^2}, \quad x > 0, \quad a > 2^{-1}$$
$$ds(x) = dx,\ dm(x) = 2\,dx,\ dk(x) = (a^2 - 2^{-2})x^{-2}\,dx.$$

We set $h(x) = \sqrt{x}K_a(x\sqrt{2\beta})$. $h \in \mathcal{H}^*_{s,m,k,\beta}$

$$\Rightarrow H^*_h\mathcal{G}_{s,m,k} = \mathcal{G}_{s_h,m_h,0} = \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{1}{2x} + \frac{\sqrt{2\beta}K'_a(x\sqrt{2\beta})}{K_a(x\sqrt{2\beta})}\right)\frac{d}{dx},$$
$$ds_h(x) = \frac{dx}{xK_a^2(x\sqrt{2\beta})}, \quad dm_h(x) = 2xK_a^2(x\sqrt{2\beta})\,dx.$$

Remark.

0 : natural $\rightarrow$ exit