Feller property and Limit theorem of skew product diffusions

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Contents

1. DF of ODGDP

$[R_t, P^R]$: one-dim generalized diffusion process on $I = (l_1, l_2)$
s^R : scale funcition
m^R : speed measure

We consider

$$\mathcal{E}^R(u,v) = \int_I \frac{du}{ds}\frac{dv}{ds}ds$$

$$\mathcal{F}[\mathcal{E}^R] = \{u \in L^2(I; dm^R) |$$

$$u: \text{ absolutely conti. on } I \text{ w. r. t. } ds \text{ and } \mathcal{E}^R(u,u) < \infty \}$$

PROPOSITION 1. $(\mathcal{E}^R, \mathcal{F}[\mathcal{E}^R])$ *is a regular strong local irreducible Dirichlet form on* $L^2(I; dm^R)$, *Its core is*

$$\mathcal{C}^R = \{u \circ s : \ u \in C_0^1(J)\}$$

where $J = s(I)$.

Assume

Both of l_i, $i = 1, 2$, are (s^R, m^R)-entrance or natural, that is,

$$\int_{(c,l_i)} ds^R \int_{[x,c)} dm^R(y) = \infty, \ \int_{(c,l_i)} dm^R(x) \int_{[x,c)} ds^R(y) = \infty.$$

2. DF of SBM

$[\Theta_t^{(d-1)}, P^{\Theta^{(d-1)}}]$: Brownian Motion on S^{d-1}

PROPOSITION 2. *Dirichlet form of $d-1$ dim Spherical Brownian Motion is*

$$\mathcal{E}^{S^{d-1}}(u,v) = \frac{1}{2}\sum_{i=1}^{d-1}\int_{S^{d-1}} \frac{\partial u}{\partial \theta_i}\frac{\partial v}{\partial \theta_i}\frac{1}{(\sin\theta_{i-1})^2}\, dm^{(d-1)}(\theta),$$

$$\mathcal{F}[\mathcal{E}^{S^{d-1}}] = \overline{C_0^\infty((0,\pi)\times\cdots\times(0,\pi)\times S^1)},$$

where $\sin\theta_0 = 1$,

$$dm^{(d-1)}(\theta) = (\sin\theta_1)^{d-2}(\sin\theta_2)^{d-3}\cdots\sin\theta_{d-2}\, d\theta_1\cdots d\theta_{d-2}d\theta_{d-1},$$

$$\theta = (\theta_1,\theta_2,\cdots,\theta_{d-1}) \in (0,\pi)\times\cdots\times(0,\pi)\times[0,2\pi].$$

3. skew product of ODGDP and SBM

$X = [(R(t), \Theta(\mathbf{f}_t)) : P_r^R \otimes P_\theta^\Theta]$: skew product difusion,

$$\mathbf{f}_t = \int_I l^R(t,\xi) d\nu(\xi), \quad t > 0.$$

PROPOSITION 3. *Dirichlet form is*

$$\mathcal{E}^X(u,v) = \int_{S^{d-1}} \mathcal{E}^R(u(\cdot,\theta), v(\cdot,\theta))\, dm^{(d-1)}(\theta) + \int_I \mathcal{E}^{S^{d-1}}(u(r,\cdot), v(r,\cdot)) d\nu(r),$$

for $f, g \in \mathcal{C}^X$*, where* $\mathcal{C}^X = \{f(s^R(r),\theta) : f \in C_0^\infty(J \times S^{d-1})\}$ *and* $J = s^R(I)$*.*

4. FP of skew product diffusion

We set

$$p_t^X f(r,\theta) = E_r^R \otimes E_\theta^\Theta [f(R(t), \Theta(\mathbf{f}(t))].$$

$X = [(R(t), \Theta(\mathbf{f}(t)) : P_r^R \otimes P_\theta^\Theta]$ satisfies Feller property in the following sence.

THEOREM 4. *Let* $i = 1,\ 2, t > 0$ *and* $f \in C_b(I \times S^{d-1})$*.*

(i) $p_t^X f(r,\theta) \in C_b(I \times S^{d-1})$

(ii) *Assume that* l_i *is* (s^R, m^R)*-entrance and*

$$\int_{l_i}^{c} s(\xi)\nu(d\xi) = -\infty.$$

Further assume that there exists $\lim_{r\to l_i} f(r,\theta)$*. Then there exists*

$$\lim_{r\to l_i} p_t^X f(r,\theta) = \frac{1}{A_{d-1}} \int_{S^{d-1}} E_{l_i}^R[f(R_t,\xi)]\, dm^{(d-1)}(\xi),$$

where $A_{d-1} = \int_{S^{d-1}} dm^{(d-1)}$*.*

(iii) *Assume that* l_i *is* (s^R, m^R)*-natural and* $\lim_{r\to l_i} \sup_{\theta\in S^{d-1}} |f(r,\theta)| = 0$*. Then*

$$\lim_{r\to l_i} p_t^X f(r,\theta) = 0.$$

5. FP of time changed process

We consider

$$Y = \left[\left(R_{\tau_t}, \Theta_{\mathbf{f}(\tau_t)}\right), P_r^R \times P_\theta^\Theta\right].$$

τ_t : inverse of $A(t) = \int_I l^R(t,r)d\mu(r), \quad t \geq 0$,
supp$[\mu] = \Lambda$ We set

$$p_t^Y f(r,\theta) = E_r^R \otimes E_\theta^\Theta[f(R(\tau_t), \Theta(\mathbf{f}(\tau_t))].$$

THEOREM 5. *Let $i = 1,\ 2,\ t > 0$ and $f \in C_b(\Gamma)$, whrere $\Gamma = \Lambda \times S^{d-1}$.*

(i) $p_t^Y f(r,\theta) \in C_b(\Gamma)$

(ii) *Assume that l_i is (s^R, μ)-regular or exit, then*

$$\lim_{r \to l_i} p_t^Y f(r,\theta) = 0.$$

(iii) *Assume that l_i is (s^R, μ)-entrance,*

$$\int_{l_i}^{c} s^R(\xi)\nu(d\xi) = -\infty.$$

and there exists $\lim_{r \to l_i} f(r, \theta)$. Then there exists

$$\lim_{r \to l_i} p_t^X f(r, \theta) = \frac{1}{A_{d-1}} \int_{S^{d-1}} E_{l_i}^R[f(R_{\tau_t}, \xi)]\, dm^{(d-1)}(\xi).$$

(iv) *Assume that l_i is (s^R, μ)-natural and*

$$\lim_{r \to l_i} \sup_{\theta \in S^{d-1}} |f(r, \theta)| = 0.$$

Then

$$\lim_{r \to l_i} p_t^Y f(r, \theta) = 0.$$

6. DF of time changed process

Assume $\Lambda \neq I$, $\nu = m^R$ on $I \setminus \Lambda$, and for any compact set $B \subset I$, there exist a positive constant M_B satisfying

$$1_B(r)\, ds^R \leq M_B 1_B(r) dm^R.$$

We set $I \setminus \Lambda = \cup_{k \in K}(a_k, b_k)$. By virtue of a general theory of ODGDP's, there exist the following limits.

$$J_k^{1,1}(\theta,\varphi) := \lim_{r \downarrow a_k} D_{s^R(r)} G_{k,2}(r;\theta,\varphi).$$

$$J_k^{1,2}(\theta,\varphi) := \lim_{r \downarrow a_k} D_{s^R(r)} G_{k,1}(r;\theta,\varphi).$$

$$J_k^{2,1}(\theta,\varphi) := -\lim_{r \uparrow b_k} D_{s^R(r)} G_{k,2}(r;\theta,\varphi).$$

$$J_k^{2,2}(\theta,\varphi) := -\lim_{r \uparrow b_k} D_{s^R(r)} G_{k,1}(r;\theta,\varphi).$$

$$G_{k,1}(r;\theta,\varphi) = \int_0^\infty p^\Theta(t,\theta,\varphi) h_{I_k}^R(t,r,b_k)\, dt$$

$$= \sum_{n=0}^{\infty} \sum_{l=1}^{\kappa(n)} S_n^l(\theta) S_n^l(\varphi) E^{P_r^R} \left[e^{-\gamma_n \sigma_{b_k}^R}; \sigma_{b_k}^R < \sigma_{a_k}^R \right],$$

$$G_{k,2}(r;\theta,\varphi) = \int_0^{\infty} p^{\Theta}(t,\theta,\varphi) h_{I_k}^R(t,r,a_k)\, dt$$

$$= \sum_{n=0}^{\infty} \sum_{l=1}^{\kappa(n)} S_n^l(\theta) S_n^l(\varphi) E^{P_r^R} \left[e^{-\gamma_n \sigma_{a_k}^R}; \sigma_{a_k}^R < \sigma_{b_k}^R \right],$$

THEOREM 6. *the Dirichlet form $(\mathcal{E}^{\mathsf{Y}}, \mathcal{F}^{\mathsf{Y}})$ of Y is regular on $L^2(\Gamma, \mu \otimes m^{(d-1)})$ and has $\mathcal{C}^{\mathsf{X}}|_{\Gamma}$ as a core. For $f \in \mathcal{C}^{\mathsf{X}}|_{\Gamma}$, the Dirichlet form $(\mathcal{E}^{\mathsf{Y}}, \mathcal{F}^{\mathsf{Y}})$ is given by the following.*

$$\mathcal{E}^{\mathsf{Y}}(f,f) = \int_{\Gamma} \partial_{s^R}^* f(r,\theta)^2\, ds^R(r)\, dm^{(d-1)}(\theta) + \int_{\Lambda} \mathcal{E}^{\Theta}(f(r,\cdot), f(r,\cdot))\, d\nu(r)$$

$$+ \frac{1}{2} \sum_{k\in K: l_1 < a_k < b_k \le l_2} \int_{S^{d-1}\times S^{d-1}} \{f(a_k,\theta) - f(a_k,\varphi)\}^2 J_k^{1,1}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi)$$

$$+\frac{1}{2}\sum_{k\in K:l_1\le a_k<b_k<l_2}\int_{S^{d-1}\times S^{d-1}}\{f(b_k,\theta)-f(b_k,\varphi)\}^2 J_k^{2,2}(\theta,\varphi)\,d\mathcal{M}(\theta,\varphi)$$
$$+\frac{1}{2}\sum_{k\in K:l_1<a_k<b_k<l_2}\int_{S^{d-1}\times S^{d-1}}\{f(a_k,\theta)-f(b_k,\varphi)\}^2 J_k^{1,2}(\theta,\varphi)\,d\mathcal{M}(\theta,\varphi)$$
$$+\frac{1}{2}\sum_{k\in K:l_1<a_k<b_k\le l_2}\int_{S^{d-1}\times S^{d-1}}\{f(b_k,\theta)-f(a_k,\varphi)\}^2 J_k^{2,1}(\theta,\varphi)\,d\mathcal{M}(\theta,\varphi)$$
$$+I_1(f)+I_2(f). \qquad (1)$$

Here the first term of the right hand side vanishes in case that $\int_\Lambda ds^R(r)=0$*. The last two terms* $I_i(f),\ i=1,2$ *should be read as*

$$I_1(f)=\begin{cases}\dfrac{1}{s^R(b_k)-s^R(l_1)}\displaystyle\int_{S^{d-1}}f(b_k,\theta)^2\,dm_{d-1}^{(2)}(\theta) & \\ \qquad\qquad \text{if } l_1=a_k<b_k<l_2 \text{ and } s^R(l_1)>-\infty, & \\ 0 \qquad\qquad \text{otherwise,} & \end{cases}$$

$$I_2(f) = \begin{cases} \dfrac{1}{s^R(l_2) - s^R(a_k)} \displaystyle\int_{S^{d-1}} f(a_k,\theta)^2 \, dm^{(2)}_{d-1}(\theta) \\ \qquad\qquad \textit{if } l_1 < a_k < b_k = l_2 \textit{ and } s^R(l_2) < \infty, \\ 0 \qquad\qquad \textit{otherwise}, \end{cases}$$

EXAMPLE 7. R: Bessel process on $I = (0,\infty)$

$$\mathcal{G}^{\mathsf{R}} = \frac{1}{2}\Big(\frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr}\Big),\ ds^{\mathsf{R}}(r) = 2r^{1-d}dr,\ dm^{\mathsf{R}}(r) = r^{d-1}dr.$$

0 : $(s^{\mathsf{R}}, m^{\mathsf{R}})$-entrance. ∞: $(s^{\mathsf{R}}, m^{\mathsf{R}})$-natural

(i) $d\mu(r) = 1_{(0,a)}(r)dm^{\mathsf{R}}(r)$,

$$d\nu(r) = 1_{(0,a)}(r)d\omega(r) + 1_{(a,\infty)}(r)dm^{\mathsf{R}}(r),$$

$$(0 < a < \infty, \mathsf{supp}[\omega] = I and |\int_0^a s^{\mathsf{R}}(r)\, d\omega(r)| = \infty).$$

For $f \in C^{\mathsf{X}}|_{(0,a)\times S^{d-1}}$,

$$\begin{aligned}\mathcal{E}^{\mathsf{Y}}(f,f) =& \frac{1}{2}\int_{(0,a)\times S^{d-1}} \frac{\partial f}{\partial r}(r,\theta)^2\, r^{d-1} dr\, dm^{(d-1)}(\theta) \\ &+ \int_{(0,a)} \mathcal{E}^{\Theta}(f(r,\cdot), f(r,\cdot))\, d\omega(r) \\ &+ \frac{1}{2}\int_{S^{d-1}\times S^{d-1}} \{f(a,\theta)-f(a,\varphi)\}^2 J(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) + I(f),\end{aligned}$$

where

$$I(f) = \begin{cases} \dfrac{d-2}{2} a^{d-2} \displaystyle\int_{S^{d-1}} f(a,\theta)^2\, dm^{(d-1)}(\theta), & \text{if } d \geq 3, \\ 0, & \text{if } d = 2. \end{cases}$$

Further $J(\theta,\varphi)$ is given as follows.

$$J(\theta,\varphi) = \lim_{r\downarrow a} D_{s^{\mathsf{R}}(r)} \sum_{n=0}^{\infty} \sum_{l=1}^{\kappa(n)} S_n^l(\theta) S_n^l(\varphi) E^{P_r^{\mathsf{R}}}\left[e^{-\gamma_n \sigma_a^{\mathsf{R}}}\right]$$

Especially, if $d = 2$, then

$$J(\theta, \varphi) = \left(8\pi \sin^2 \frac{\theta - \varphi}{2}\right)^{-1}.$$

Therefore $\mathcal{E}^{\mathsf{Y}}$ corresponding to the case $d = 2$ is

$$\begin{aligned}\mathcal{E}^{\mathsf{Y}}(f, f) =& \frac{1}{2}\int_{(0,a)\times S^1} \frac{\partial f}{\partial r}(r,\theta)^2\, rdr\, d\theta + \frac{1}{2}\int_{(0,a)\times S^1} \frac{\partial f}{\partial \theta}(r,\theta)^2\, d\omega(r)\, d\theta \\ &+ \frac{1}{16\pi}\int_{S^1\times S^1} \{f(a,\theta) - f(a,\varphi)\}^2 \frac{1}{\sin^2((\theta-\varphi)/2)}\, d\theta d\varphi.\end{aligned}$$

(ii) $d\mu(r) = 1_{(a,b)}(r) dm^{rmR}(r)$,

$$d\nu(r) = 1_{(0,a)\cup(b,\infty)}(r) dm^{\mathsf{R}}(r) + 1_{(a,b)}(r) d\omega(r),$$
$$(0 < a < b < \infty, \mathrm{supp}[\omega] = I).$$

For $f \in C^{\mathsf{X}}|_{(a,b)\times S^{d-1}}$,

$$\begin{aligned}\mathcal{E}^{\mathsf{Y}}(f,f) =& \frac{1}{2}\int_{(a,b)\times S^{d-1}} \frac{\partial f}{\partial r}(r,\theta)^2\, r^{d-1} dr\, dm^{(d-1)}(\theta) \\ &+ \int_{(a,b)} \mathcal{E}^{\Theta}(f(r,\cdot), f(r,\cdot))\, d\omega(r) \\ &+ \frac{1}{2}\int_{S^{d-1}\times S^{d-1}} \{f(a,\theta) - f(a,\varphi)\}^2 J_1(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) \\ &+ \frac{1}{2}\int_{S^{d-1}\times S^{d-1}} \{f(b,\theta) - f(b,\varphi)\}^2 J_2(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) + I(f),\end{aligned}$$

where

$$I(f) = \begin{cases} \dfrac{d-2}{2} b^{d-2} \displaystyle\int_{S^{d-1}} f(b,\theta)^2 \, dm^{(d-1)}(\theta), & \text{if } d \geq 3, \\ 0, & \text{if } d = 2. \end{cases}$$

Further $J(\theta, \varphi)_i$, $i = 1, 2$, is given as follows.

$$J_1(\theta, \varphi) = - \lim_{r \uparrow a} D_{s^{\mathsf{R}}(r)} \sum_{n=0}^{\infty} (r/a)^n \sum_{l=1}^{\kappa(n)} S_n^l(\theta) S_n^l(\varphi),$$

$$J_2(\theta, \varphi) = \lim_{r \downarrow b} D_{s^{\mathsf{R}}(r)} \sum_{n=0}^{\infty} (r/a)^n \sum_{l=1}^{\kappa(n)} S_n^l(\theta) S_n^l(\varphi)$$

When $d = 2$,

$$J_1(\theta,\varphi) = \left(8\pi \sin^2 \frac{\theta-\varphi}{2}\right)^{-1}, \quad J_2(\theta,\varphi) = \left(8\pi \sin^2 \frac{\theta-\varphi}{2}\right)^{-1}.$$

Therefore $\mathcal{E}^{\mathsf{Y}}$ corresponding to the case $d = 2$ is

$$\begin{aligned}\mathcal{E}^{\mathsf{Y}}(f,f) =& \frac{1}{2}\int_{(a,b)\times S^1} \frac{\partial f}{\partial r}(r,\theta)^2\, r dr\, d\theta + \frac{1}{2}\int_{(a,b)\times S^1} \frac{\partial f}{\partial \theta}(r,\theta)^2\, d\omega(r)\, d\theta \\ &+ \frac{1}{16\pi}\int_{S^1\times S^1} \{f(a,\theta) - f(a,\varphi)\}^2 \frac{1}{\sin^2((\theta-\varphi)/2)}\, d\theta d\varphi \\ &+ \frac{1}{16\pi}\int_{S^1\times S^1} \{f(b,\theta) - f(b,\varphi)\}^2 \frac{1}{\sin^2((\theta-\varphi)/2)}\, d\theta d\varphi.\end{aligned}$$